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Vectors:

A Curriculum Module for AP® Calculus BC

2010
Curriculum Module

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Contents

Vectors in AP® Calculus BC

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Introduction

According to the *AP® Calculus BC Course Description*, students in Calculus BC are required to know:

- Analysis of planar curves given in parametric form and vector form, including velocity and acceleration vectors
- Derivatives of parametric and vector functions
- The length of a curve, including a curve given in parametric form

What does this mean? For parametric equations $x = f(t)$ and $y = g(t)$, students should be able to:

1. Sketch the curve defined by the parametric equations and eliminate the parameter.
2. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and evaluate them for a given value of t.
3. Write an equation for the tangent line to the curve for a given value of t.
4. Find the points of horizontal and vertical tangency.
5. Find the length of an arc of a curve given by parametric equations.

For vectors describing particle motion along a curve in terms of a time variable t, students should be able to:

1. Find the velocity and acceleration vectors when given the position vector.
2. Given the components of the velocity vector and the position of the particle at a particular value of t, find the position at another value of t.
3. Given the components of the acceleration vector and the velocity of the particle at a particular value of t, find the velocity at another value of t.
4. Find the slope of the path of the particle for a given value of t.
5. Write an equation for the tangent line to the curve for a given value of t.
6. Find the values of t at which the line tangent to the path of the particle is horizontal or vertical.

7. Find the speed of the particle (sometimes asked as the magnitude of the velocity vector) at a given value of t.
8. Find the distance traveled by the particle for a given interval of time.

I like to start this unit with parametric equations, teaching the students the five types of parametric problems listed above. Then I take a day to review the concept of motion along a horizontal or vertical line, which they learned earlier in the year, as a bridge to motion along a curve.

The unit on parametric equations and vectors takes me six days to cover (see the following schedule), not including a test day. I teach on a traditional seven-period day, with 50 minutes in each class period.

Day 1 — Graphing parametric equations and eliminating the parameter

Day 2 — Calculus of parametric equations: Finding $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and evaluating them for a given value of t, finding points of horizontal and vertical tangency, finding the length of an arc of a curve

Day 3 — Review of motion along a horizontal and vertical line. (The students have studied this topic earlier in the year.)

Days 4, 5 and 6 — Particle motion along a curve (vectors):

- Finding the velocity and acceleration vectors when given the position vector;
- Given the components of the velocity vector and the position of the particle at one value of t, finding the position of the particle at a different value of t;
- Finding the slope of the path of the particle for a given value of t;
- Writing an equation for the tangent line to the curve for a given value of t;
- Finding the values of t at which the line tangent to the path of the particle is horizontal or vertical;
- Finding the speed of the particle; and
- Finding the distance traveled by the particle for a given interval of time.

Day 1: Graphing Parametric Equations and Eliminating the Parameter

My students have studied parametric equations and vectors in their precalculus course, so this lesson is a review for them. Many of them have also studied parametric equations and vectors in their physics course. If your textbook contains this material, you might want to follow your book here.

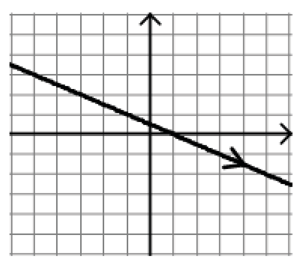
Directions: Make a table of values and sketch the curve, indicating the direction of your graph. Then eliminate the parameter.

(a) $x = 2t - 1$ and $y = 1 - t$

Solution: First make a table using various values of *t*, including negative numbers, positive numbers and zero, and determine the *x* and *y* values that correspond to these *t* values.

t	–2	–1	0	1	2	3
x	–5	–3	–1	1	3	5
y	3	2	1	0	–1	–2

Plot the ordered pairs (x, y), drawing an arrow on the graph to indicate its direction as *t* increases.



To eliminate the parameter, solve $x = 2t - 1$ for $t = \frac{x+1}{2}$ or $t = \frac{1}{2}x + \frac{1}{2}$. Then

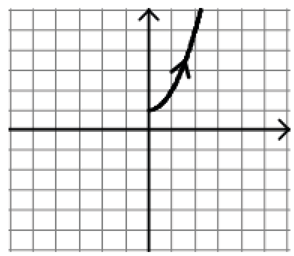
substitute $t = \frac{1}{2}x + \frac{1}{2}$ in place of *t* in the equation $y = 1 - t$ to get $y = -\frac{1}{2}x + \frac{1}{2}$.

Look at the graph of the parametric equations to see if this equation matches the graph, and observe that it does.

(b) $x = \sqrt{t},\ y = t + 1$

Solution: Since $x = \sqrt{t}$, we can use only nonnegative values for *t*.

t	0	1	4	9
x	0	1	2	3
y	1	2	5	10

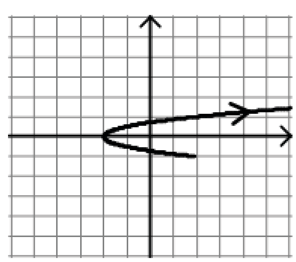


To eliminate the parameter, solve $x = \sqrt{t}$ for $t = x^2$. Then substitute $t = x^2$ into *y*'s equation so that $y = x^2 + 1$. To make this equation match the graph, we must restrict *x* so that it is greater than or equal to 0. The solution is $y = x^2 + 1,\ x \geq 0$.

(c) $x = t^2 - 2$ and $y = \frac{t}{2}, -2 \leq t \leq 3$

Solution: First make a table using *t* values that lie between –2 and 3, and determine the *x* and *y* values that correspond to these *t* values.

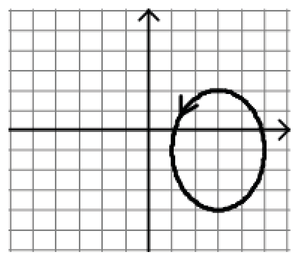
t	–2	–1	0	1	2	3
x	2	–1	–2	–1	2	7
y	–1	$\frac{-1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$



To eliminate the parameter, solve $y = \frac{t}{2}$ for *t* to find that $t = 2y$, $-1 \leq y \leq \frac{3}{2}$. Then substitute 2*y* in place of *t* in the other equation so that $x = 4y^2 - 2$. To make this equation match the graph, we must restrict y so that it lies between –1 and $\frac{3}{2}$. The solution is $x = 4y^2 - 2, -1 \le y \le \frac{3}{2}$.

(d) $x = 3 + 2\cos t,\ y = -1 + 3\sin t$

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	5	3	1	3	5
y	–1	2	–1	–4	–1



Solution: To eliminate the parameter, solve for $\cos t$ in x's equation to get $\cos t = \frac{x-3}{2}$ and $\sin t$ in y's equation to get $\sin t = \frac{y+1}{3}$. Substitute into the trigonometric identity $\cos^2 t + \sin^2 t = 1$ to get $\frac{(x-3)^2}{4} + \frac{(y+1)^2}{9} = 1$. Discuss with the students the fact that this is an ellipse centered at the point (3, –1) with a horizontal axis of length 4 and a vertical axis of length 6.

Day 1 Homework

Make a table of values and sketch the curve, indicating the direction of your graph. Then eliminate the parameter. Do not use your calculator.

1. $x = 2t + 1$ and $y = t - 1$

2. $x = 2t$ and $y = t^2, -1 \le t \le 2$

3. $x = 2 - t^2$ and $y = t$

4. $x = \sqrt{t}$ and $y = t - 3$

5. $x = t - 2$ and $y = 1 - \sqrt{t}$

6. $x = 2t$ and $y = |t - 1|$

7. $x = t$ and $y = \dfrac{1}{t^2}$

8. $x = 2\cos t - 1$ and $y = 3\sin t + 1$

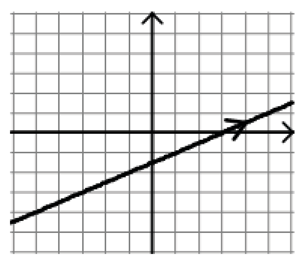
9. $x = 2\sin t - 1$ and $y = \cos t + 2$

10. $x = \sec t$ and $y = \tan t$

Answers to Day 1 Homework

1. $x = 2t + 1$ and $y = t - 1$

t	–2	–1	0	1	2
x	–3	–1	1	3	5
y	–3	–2	–1	0	1



To eliminate the parameter, solve for $t = \frac{1}{2}x - \frac{1}{2}$. Substitute into y's equation to get $y = \frac{1}{2}x - \frac{3}{2}$.

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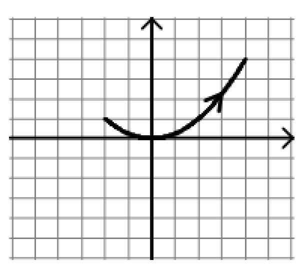
2. $x = 2t$ and $y = t^2, -1 \le t \le 2$

t	–1	0	1	2
x	–2	0	2	4
y	1	0	1	4



To eliminate the parameter, solve for $t = \frac{x}{2}$. Substitute into y's equation to get $y = \frac{x^2}{4}, -2 \le x \le 4$. (Note: The restriction on x is needed for the graph of $y = \frac{x^2}{4}$ to match the parametric graph.)

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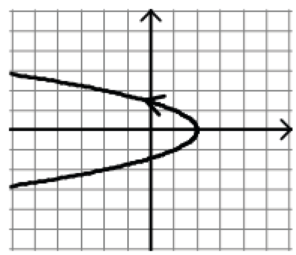
3. $x = 2 - t^2$ and $y = t$

t	–2	–1	0	1	2
x	–2	1	2	1	–2
y	–2	–1	0	1	2



To eliminate the parameter, notice that $t = y$. Substitute into x's equation to get $x = 2 - y^2$.

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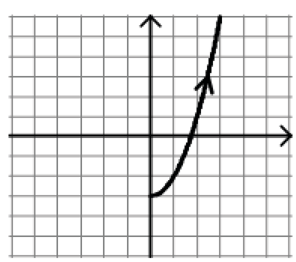
4. $x = 2 - t^2$ and $y = t - 3$

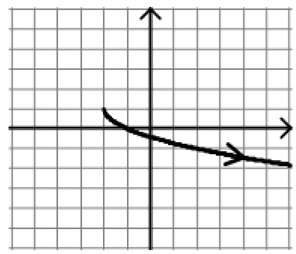
t	0	1	4	9
x	0	1	2	3
y	–3	–2	1	6



To eliminate the parameter, solve for $t = x^2$. Substitute into y's equation to get $y = x^2 - 3, \ge 0$. (Note: The restriction on x is needed for the graph of $y = x^2 - 3$ to match the parametric graph.)

5. $x = t - 2$ and $y = 1 - \sqrt{t}$

t	0	1	4	9
x	–2	–1	2	7
y	1	0	–1	–2



To eliminate the parameter, solve for $t = x + 2$, $x \geq -2$ (since $t \geq 0$). Substitute into y's equation to get $y = 1 - \sqrt{x+2}$.

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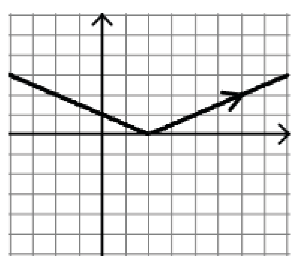
6. $x = 2t$ and $y = |t - 1|$

t	–2	–1	0	1	2	3
x	–4	–2	0	2	4	6
y	3	2	1	0	1	2

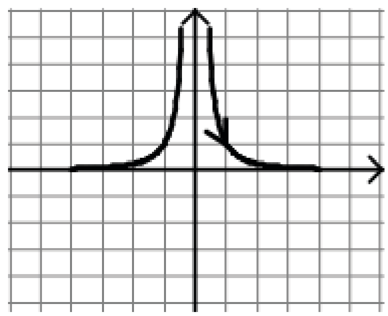


To eliminate the parameter, solve for $t = \dfrac{x}{2}$. Substitute into y's equation to get

$y = \left|\dfrac{x}{2} - 1\right|$ or $y = \dfrac{|x-2|}{2}$.

7. $x = t$ and $y = \dfrac{1}{t^2}$

t	–2	–1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
x	–2	–1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	$\frac{1}{4}$	1	4	und.	4	1	$\frac{1}{4}$



To eliminate the parameter, notice that $t = x$. Substitute into y's equation to get

$y = \dfrac{1}{x^2}$.

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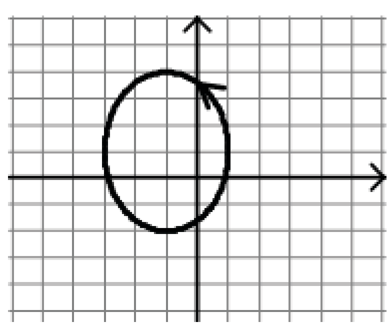
8. $x = 2\cos t - 1$ and $y = 3\sin t + 1$

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	1	-1	-3	-1	1
y	1	4	1	-2	1

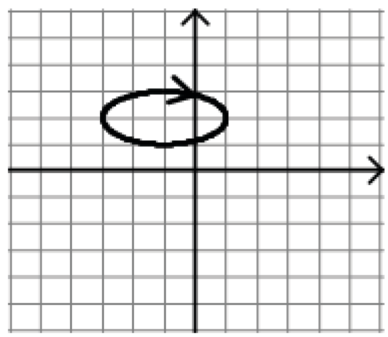


To eliminate the parameter, solve for $\cos t$ in x's equation and $\sin t$ in y's equation. Substitute into the trigonometric identity $\cos^2 t + \sin^2 t = 1$ to get

$\dfrac{(x+1)^2}{4} + \dfrac{(y-1)^2}{9} = 1$.

9. $x = 2\sin t - 1$ and $y = \cos t + 2$

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{4}$	2π
x	–1	1	–1	–3	–1
y	3	2	1	2	3



To eliminate the parameter, solve for cos t in y's equation and sin t in x's equation. Substitute into the trigonometric identity $\cos^2 t + \sin^2 t = 1$ to get

$$\frac{(x+1)^2}{4} + (y-2)^2 = 1.$$

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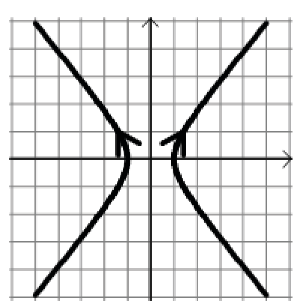
10. $x = \sec t$ and $y = \tan t$

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
x	1	$\sqrt{2}$	und.	$-\sqrt{2}$	–1	$-\sqrt{2}$	und.	$\sqrt{2}$	1
y	0	1	und.	–1	0	1	und.	–1	0



To eliminate the parameter, substitute into the trigonometric identity $1 + \tan^2 t = \sec^2 t$ to get $1 + y^2 = x^2$ or $x^2 - y^2 = 1$.

Day 2: Parametric Equations and Calculus

If your textbook contains this material, you might want to follow your book here.

Formulas to Know

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at the point (x, y) is given by $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, where $\frac{dx}{dt} \neq 0$, and the second derivative is given by

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$. The reason that $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, where $\frac{dx}{dt} \neq 0$, is the Chain Rule:

Since y is a function of x, and x is a function of t, the Chain Rule gives $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$,

hence $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, where $\frac{dx}{dt} \neq 0$. Applying this formula to $\frac{dy}{dx}$ and x rather than to y and x, we have $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$.

Or, applying the Chain Rule: Since $\frac{dy}{dx}$ is a function of x, and x is a function of t, we have $\frac{d}{dt}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{dy}{dx}\right] \cdot \frac{dx}{dt} = \frac{d^2y}{dx^2} \cdot \frac{dx}{dt}$, hence $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$.

Example 1 (no calculator): Given the parametric equations $x = 2\sqrt{t}$ and $y = 3t^2 - 2t$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution:

To find $\frac{dy}{dx}$, we must differentiate both of the parametric equations with respect to t.

Since $\frac{dy}{dt} = \frac{d}{dt}\left[3t^2 - 2t\right] = 6t - 2$ and $\frac{dx}{dt} = \frac{d}{dt}\left[2\sqrt{t}\right] = 2 \cdot \frac{1}{2}t^{-\frac{1}{2}} = t^{-\frac{1}{2}}$, then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t-2}{t^{-1/2}} = 6t^{3/2} - 2t^{1/2}.$$

To find $\frac{d^2y}{dx^2}$, we must differentiate $\frac{dy}{dx}$ with respect to t and divide by $\frac{dx}{dt}$:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[6t^{3/2} - 2t^{1/2}\right]}{\frac{dx}{dt}} = \frac{9t^{1/2} - t^{-1/2}}{\frac{dx}{dt}} = \frac{9t^{1/2} - t^{-1/2}}{t^{-1/2}} = 9t - 1.$$

Example 2 (no calculator): Given the parametric equations $x = 4\cos t$ and $y = 3\sin t$, write an equation of the tangent line to the curve at the point where $t = \frac{3\pi}{4}$.

Solution:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[3\sin t\right]}{\frac{d}{dt}\left[4\cos t\right]} = \frac{3\cos t}{-4\sin t} = -\frac{3}{4}\cot t.$$

When $t = \frac{3\pi}{4}$, $\frac{dy}{dx} = -\frac{3}{4}(-1) = \frac{3}{4}$, $x = 4\left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$ and $y = 3\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}$.

Therefore the tangent line equation is: $y - \frac{3\sqrt{2}}{2} = \frac{3}{4}\left(x + 2\sqrt{2}\right)$.

(Remind students that they may leave their tangent line equations in point-slope form.)

Example 3 (no calculator): Find all points of horizontal and vertical tangency given the parametric equations $x = t^2 + t,\ y = t^2 - 3t + 5$.

Solution:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[t^2 - 3t + 5\right]}{\frac{d}{dt}\left[t^2 + t\right]} = \frac{2t - 3}{2t + 1}.$$

A horizontal tangent will occur when $\frac{dy}{dx} = 0$, which happens when $2t - 3 = 0$ (and $2t + 1 \neq 0$), so a horizontal tangent occurs at $t = \frac{3}{2}$. Substituting $t = \frac{3}{2}$ into the given equations, we find that a horizontal tangent will occur at $\left(\frac{15}{4}, \frac{11}{4}\right)$. A vertical tangent will occur when $\frac{dy}{dx}$ is undefined, which happens when $2t + 1 = 0$ (and $2t - 3 \neq 0$), so a vertical tangent occurs at $t = -\frac{1}{2}$. Substituting $t = -\frac{1}{2}$ into the given equations, we find that a vertical tangent will occur at $\left(-\frac{1}{4}, \frac{27}{4}\right)$.

Example 4 (no calculator): Set up an integral expression for the arc length of the curve given by the parametric equations $x = t^2 + 1$, $y = 4t^3 - 1$, $0 \le t \le 1$. Do not evaluate.

Solution:

For parametric equations $x = f(t)$ and $y = g(t)$, $a \le t \le b$, the formula for arc length is:

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

For our problem, $\frac{dx}{dt} = \frac{d}{dt}\left[t^2 + 1\right] = 2t$ and $\frac{dy}{dt} = \frac{d}{dt}\left[4t^3 - 1\right] = 12t^2$, so the arc length is given by the integral expression $s = \int_0^1 \sqrt{(2t)^2 + \left(12t^2\right)^2}\, dt = \int_0^1 \sqrt{4t^2 + 144t^4}\, dt$.

Day 2 Homework

Do not use your calculator on the following problems.

On problems 1–5, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

1. $x = t^2,\ y = t^2 + 6t + 5$

2. $x = t^2 + 1,\ y = 2t^3 - t^2$

3. $x = \sqrt{t},\ y = 3t^2 + 2t$

4. $x = \ln t,\ y = t^2 + t$

5. $x = 3\sin t + 2,\ y = 4\cos t - 1$

6. A curve C is defined by the parametric equations $x = t^2 + t - 1,\ y = t^3 - t^2$. Find:

(a) $\frac{dy}{dx}$ in terms of t.

(b) an equation of the tangent line to C at the point where $t = 2$.

7. A curve C is defined by the parametric equations $x = 2\cos t,\ y = 3\sin t$. Find:

(a) $\frac{dy}{dx}$ in terms of t.

(b) an equation of the tangent line to C at the point where $t = \frac{\pi}{4}$.

On problems 8–10, find:

(a) $\frac{dy}{dx}$ in terms of t.

(b) all points of horizontal and vertical tangency.

8. $x = t + 5,\ y = t^2 - 4t$

9. $x = t^2 - t + 1,\ y = t^3 - 3t$

10. $x = 3 + 2\cos t,\ y = -1 + 4\sin t,\ 0 \le t < 2\pi$

On problems 11–12, a curve C is defined by the parametric equations given. For each problem, write an integral expression that represents the length of the arc of the curve over the given interval.

11. $x = t^2,\ y = t^3,\ 0 \le t \le 2$

12. $x = e^{2t} + 1,\ y = 3t - 1,\ -2 \le t \le 2$

Answers to Day 2 Homework

1. Since $\dfrac{dy}{dt}=\dfrac{d}{dt}\left[t^2+6t+5\right]=2t+6$ and $\dfrac{dx}{dt}=\dfrac{d}{dt}\left[t^2\right]=2t$,

then $\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=\dfrac{2t+6}{2t}=1+\dfrac{3}{t}$.

To find $\dfrac{d^2y}{dx^2}$, we must differentiate $\dfrac{dy}{dx}$ with respect to t and divide by $\dfrac{dx}{dt}$:

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}=\frac{\frac{d}{dt}\left[1+\frac{3}{t}\right]}{\frac{dx}{dt}}=\frac{-\frac{3}{t^2}}{\frac{dx}{dt}}=\frac{-\frac{3}{t^2}}{2t}=-\frac{3}{2t^3}$$

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2. $\dfrac{dy}{dt}=\dfrac{d}{dt}\left[2t^3-t^2\right]=6t^2-2t$ and $\dfrac{dx}{dt}=\dfrac{d}{dt}\left[t^2+1\right]=2t$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{6t^2-2t}{2t}=3t-1$$

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}=\frac{\frac{d}{dt}\left[3t-1\right]}{\frac{dx}{dt}}=\frac{3}{\frac{dx}{dt}}=\frac{3}{2t}$$

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3. $\dfrac{dy}{dt}=\dfrac{d}{dt}\left[3t^2+2t\right]=6t+2$ and $\dfrac{dx}{dt}=\dfrac{d}{dt}\left[\sqrt{t}\right]=\dfrac{1}{2}t^{-\frac{1}{2}}$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{6t+2}{\frac{1}{2}t^{-\frac{1}{2}}}=12t^{\frac{3}{2}}+4t^{\frac{1}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}=\frac{\frac{d}{dt}\left[12t^{\frac{3}{2}}+4t^{\frac{1}{2}}\right]}{\frac{dx}{dt}}=\frac{18t^{\frac{1}{2}}+2t^{-\frac{1}{2}}}{\frac{dx}{dt}}=\frac{18t^{\frac{1}{2}}+2t^{-\frac{1}{2}}}{\frac{1}{2}t^{-\frac{1}{2}}}=36t+4$$

4. $\dfrac{dy}{dt} = \dfrac{d}{dt}\left[t^2 + t\right] = 2t + 1$ and $\dfrac{dx}{dt} = \dfrac{d}{dt}\left[\ln t\right] = \dfrac{1}{t}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+1}{\frac{1}{t}} = 2t^2 + t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[2t^2 + t\right]}{\frac{dx}{dt}} = \frac{4t+1}{\frac{dx}{dt}} = \frac{4t+1}{\frac{1}{t}} = 4t^2 + t$$

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5. $\dfrac{dy}{dt} = \dfrac{d}{dt}\left[4\cos t - 1\right] = -4\sin t$ and $\dfrac{dx}{dt} = \dfrac{d}{dt}\left[3\sin t + 2\right] = 3\cos t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-4\sin t}{3\cos t} = -\frac{4}{3}\tan t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[-\frac{4}{3}\tan t\right]}{\frac{dx}{dt}} = \frac{-\frac{4}{3}\sec^2 t}{\frac{dx}{dt}} = \frac{-\frac{4}{3}\sec^2 t}{3\cos t} = -\frac{4}{9}\sec^3 t$$

---

6. **(a)** $\dfrac{dy}{dx} = \dfrac{3t^2 - 2t}{2t+1}$.

(b) When $t = 2$, $\dfrac{dy}{dx} = \dfrac{3 \cdot 2^2 - 2 \cdot 2}{2 \cdot 2 + 1} = \dfrac{8}{5}$, $x = 5$ and $y = 4$, so the tangent line equation is $y - 4 = \dfrac{8}{5}(x - 5)$.

7. **(a)** $\dfrac{dy}{dx} = \dfrac{3\cos t}{-2\sin t} = -\dfrac{3}{2}\cot t$.

(b) When $t = \dfrac{\pi}{4}$, $\dfrac{dy}{dx} = -\dfrac{3}{2}\cot\dfrac{\pi}{4} = -\dfrac{3}{2}$, $x = \sqrt{2}$ and $y = \dfrac{3\sqrt{2}}{2}$, so the tangent line equation is $y - \dfrac{3\sqrt{2}}{2} = -\dfrac{3}{2}\left(x - \sqrt{2}\right)$.

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8. **(a)** $\dfrac{dy}{dx} = \dfrac{2t-4}{1}$ with $\dfrac{dy}{dt} = 2t - 4$ and $\dfrac{dx}{dt} = 1$.

(b) A horizontal tangent occurs when $\dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} \neq 0$, so a horizontal tangent occurs when $2t - 4 = 0$, or at $t = 2$. When $t = 2$, $x = 7$ and $y = -4$, so a horizontal tangent occurs at the point $(7, -4)$. A vertical tangent occurs when $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} \neq 0$.

Since $1 \neq 0$, there is no point of vertical tangency on this curve.

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9. **(a)** $\dfrac{dy}{dx} = \dfrac{3t^2 - 3}{2t - 1}$ with $\dfrac{dy}{dt} = 3t^2 - 3$ and $\dfrac{dx}{dt} = 2t - 1$.

(b) A horizontal tangent occurs when $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} \neq 0$, so a horizontal tangent occurs when $3t^2 - 3 = 0$, or when $t = \pm 1$. When $t = 1$, $x = 1$ and $y = -2$, and when $t = -1$, $x = 3$ and $y = 2$, so horizontal tangents occur at the points $(1, -2)$ and $(3, 2)$.

A vertical tangent occurs when $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} \neq 0$ so a vertical tangent occurs when $2t - 1 = 0$, or $t = \dfrac{1}{2}$. When $t = \dfrac{1}{2}$, $x = \dfrac{3}{4}$ and $y = -\dfrac{11}{8}$, so a vertical tangent occurs at the point $\left(\dfrac{3}{4}, -\dfrac{11}{8}\right)$.

10. (a) $\dfrac{dy}{dx} = \dfrac{4\cos t}{-2\sin t} = -2\cot t$ with $\dfrac{dy}{dt} = 4\cos t$ and $\dfrac{dx}{dt} = -2\sin t$.

(b) A horizontal tangent occurs when $\dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} \neq 0$, so a horizontal tangent occurs when $4\cos t = 0$, or at $t = \dfrac{\pi}{2}$ and $t = \dfrac{3\pi}{2}$. When $t = \dfrac{\pi}{2}$, $x = 3$ and $y = 3$, and when $t = \dfrac{3\pi}{2}$, $x = 3$ and $y = -5$, so horizontal tangents occur at the points $(3, 3)$ and $(3, -5)$.

A vertical tangent occurs when $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} \neq 0$, so a vertical tangent occurs when $-2\sin t = 0$, or when $t = 0$ and $t = \pi$. When $t = 0$, $x = 5$ and $y = -1$, and when $t = \pi$, $x = 1$ and $y = -1$, so vertical tangents occur at the points $(5, -1)$ and $(1, -1)$.

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11. $s = \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^2 \sqrt{(2t)^2 + (3t^2)^2}\, dt = \int_0^2 \sqrt{4t^2 + 9t^4}\, dt$.

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12. $s = \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{-2}^2 \sqrt{\left(2e^{2t}\right)^2 + (3)^2}\, dt = \int_{-2}^2 \sqrt{4e^{4t} + 9}\, dt$.

Day 3: Review of Motion Along a Line

My students study motion along a line early in the year, so this assignment is a review for them. I like to spend a day on motion along a line as a segue into motion along a curve. For an excellent introduction to motion along a line, see the Curriculum Module on motion by Dixie Ross at AP Central®. (http://apcentral.collegeboard.com/apc/public/repository/AP_CurricModCalculusMotion.pdf)

Day 3 Homework

The following problems are from old AP Exams and the sample multiple-choice problems in the Course Description, available at AP Central (http://apcentral.collegeboard.com/apc/public/courses/teachers_corner/2118.html).

Multiple-Choice Items:

1. 2003 AP Calculus AB Exam, Item 25 (no calculator):

A particle moves along the x-axis so that at time $t \geq 0$ its position is given by $x(t) = 2t^3 - 21t^2 + 72t - 53$. At what time t is the particle at rest?

(A) $t = 1$ only

(B) $t = 3$ only

(C) $t = \frac{7}{2}$ only

(D) $t = 3$ and $t = \frac{7}{2}$

(E) $t = 3$ and $t = 4$

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2. 1998 AP Calculus AB Exam, Item 24 (no calculator):
The maximum acceleration attained on the interval $0 \leq t \leq 3$ by the particle whose velocity is given by $v(t) = t^3 - 3t^2 + 12t + 4$ is

(A) 9

(B) 12

(C) 14

(D) 21

(E) 40

3. AP Calculus AB, sample multiple-choice Item 9 (no calculator):
The position of a particle moving along a line is given by
$s(t) = 2t^3 - 24t^2 + 90t + 7$ for $t \geq 0$.

For what values of t is the speed of the particle increasing?

(A) $3 < t < 4$ only

(B) $t > 4$ only

(C) $t > 5$ only

(D) $0 < t < 3$ and $t > 5$

(E) $3 < t < 4$ and $t > 5$

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4. 2003 AP Calculus AB Exam, Item 76 (calculator):
A particle moves along the x-axis so that at any time $t \geq 0$, its velocity is given by $v(t) = 3 + 4.1\cos(0.9t)$. What is the acceleration of the particle at time $t = 4$?

(A) –2.016

(B) –0.677

(C) 1.633

(D) 1.814

(E) 2.97

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5. 2003 AP Calculus AB Exam, Item 91 (calculator):
A particle moves along the x-axis so that at any time $t > 0$, its acceleration is given by $a(t) = \ln(1 + 2^t)$. If the velocity of the particle is 2 at time $t = 1$, then the velocity of the particle at time $t = 2$ is

(A) 0.462

(B) 1.609

(C) 2.555

(D) 2.886

(E) 3.346

6. AP Calculus AB, sample multiple-choice Item 19 (calculator):
Two particles start at the origin and move along the *x*-axis. For $0 \le t \le 10$, their respective position functions are given by $x_1 = \sin t$ and $x_2 = e^{-2t} - 1$. For how many values of *t* do the particles have the same velocity?

(A) None

(B) One

(C) Two

(D) Three

(E) Four

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7. AP Calculus AB, sample multiple-choice Item 15 (calculator):

A particle travels along a straight line with a velocity of $v(t) = 3e^{\left(-t/2\right)}\sin(2t)$ meters per second. What is the total distance traveled by the particle during the time interval $0 \le t \le 2$ seconds?

(A) 0.835

(B) 1.850

(C) 2.055

(D) 2.261

(E) 7.025

Free-Response Questions:

8. 2004 AP Calculus AB Exam, FRQ 3 (calculator):
A particle moves along the *y*-axis so that its velocity at time $t \ge 0$ is given by $v(t) = 1 - \tan^{-1}\left(e^{t}\right)$. At time $t = 0$, the particle is at $y = -1$. (Note: $\tan^{-1} x = \arctan x$.)

(a) Find the acceleration of the particle at time $t = 2$.

(b) Is the speed of the particle increasing or decreasing at time $t = 2$? Give a reason for your answer.

(c) Find the time $t \ge 0$ at which the particle reaches its highest point. Justify your answer.

(d) Find the position of the particle at time $t = 2$. Is the particle moving toward the origin or away from the origin at time $t = 2$? Justify your answer.

9. 2006 AP Calculus AB/BC Exams, Item 4 (no calculator):

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

(a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

(c) Rocket B is launched upward with an acceleration of $a(t) = \dfrac{3}{\sqrt{t+1}}$ feet per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

Answers to Day 3 Homework

1. Since $x'(t) = 6t^2 - 42t + 72 = 6(t^2 - 7t + 12) = 6(t - 3)(t - 4) = 0$ when $t = 3$ and when $t = 4$, the answer is E.

2. Note that $a(t) = 3t^2 - 6t + 12$, so that $a'(t) = 6t - 6 = 0$ when $t = 1$. Computing the acceleration at the critical number and at the endpoints of the interval, we have $a(0) = 12$, $a(1) = 9$, and $a(3) = 21$. The maximum acceleration is 21, so the answer is D.

3. Note that $v(t) = 6t^2 - 48t + 90 = 6(t - 3)(t - 5)$ and $a(t) = 12t - 48 = 12(t - 4)$. The speed is increasing on $3 < t < 4$, where the velocity and the acceleration are both negative, and also for $t > 5$, where the velocity and the acceleration are both positive, so the answer is E.

4. Since $\frac{d}{dt}\left[3 + 4.1\cos(0.9t)\right]_{t-4} = 1.633$, the answer is C.

5. Since $v(2) = 2 + \int_1^2 \ln(1 + 2^t)\,dt = 3.346$, the answer is E.

6. First find $\frac{d}{dt}\left[\sin t\right]=\cos t$ and $\frac{d}{dt}\left[e^{-2t}\right]=-2e^{-2t}$. Then graph $y_1 = \cos x$ and $y_2 = -2e^{-2x}$ in function mode with an x-window of [0, 10] and a y- window of [–1 , 1]. The two graphs intersect at three points, so the answer is D.

7. Distance = $\int_0^2 \left|v(t)\right| dt = \int_0^2 \left|3e^{\left(-t/2\right)}\sin(2t)\right| dt = 2.261$, so the answer is D.

8. **(a)** $a(2) = v'(2) = -0.132$ or -0.133.

(b) $v(2) = -\,0.436$. Since $a(2) < 0$, and $v(2) < 0$, the speed is increasing.

(c) Note that $v(t) = 0$ when $\tan^{-1}(e^t) = 1$. The only critical number for y is $t = \ln(\tan 1) = 0.443$. Since $v(t) > 0$ for $0 \le t < \ln(\tan 1)$ and $v(t) < 0$ for $t > \ln(\tan 1)$, $y(t)$ has an absolute maximum at $t = 0.443$.

(d) $y(2) = -1 + \int_0^2 v(t)\,dt = -1.360 \text{ or } -1.361.$

Since $v(2) < 0$ and $y(2) < 0$, the particle is moving away from the origin.

9. **(a)** Average acceleration of rocket A is

$$\frac{v(80)-v(0)}{80-0}=\frac{49-5}{80}=\frac{11}{20}\text{ ft / sec}^2.$$

(b) Since the velocity is positive, $\int_{10}^{70} v(t)\,dt$ represents the distance, in feet, traveled by rocket A from $t = 10$ seconds to $t = 70$ seconds. A midpoint Riemann sum is

$$20\left[v(20)+v(40)+v(60)\right]=20(22+35+44)=2020\text{ ft}.$$

(c) Let $v_B(t)$ be the velocity of rocket B at time t. Then $v_B(t)=\int \frac{3}{\sqrt{t+1}}\,dt = 6\sqrt{t+1}+C$. Since $2 = v_B(0) = 6 + C$, then C = – 4 and $v_B(t) = 6\sqrt{t+1}-4$. Hence, $v_B(80) = 50 > 49 = v(80)$ and Rocket B is traveling faster at time $t = 80$ seconds.

Day 4: Motion Along a Curve — Vectors

I give my students the following list of terms and formulas to know.

Parametric Equations, Vectors, and Calculus — Terms and Formulas to Know:

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at the point (x, y) is given by $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ where $\frac{dx}{dt} \neq 0$, and the second derivative is given by $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$.

The derivative $\frac{dy}{dx}$ also may be interpreted as the slope of the tangent line to the curve C, or as the slope of the path of a particle traveling along the curve C, or as the rate of change of y with respect to x.

The second derivative $\frac{d^2y}{dx^2}$ is the rate of change of the slope of the curve C with respect to x.

$x'(t) = \frac{dx}{dt}$ is the rate at which the x-coordinate is changing with respect to t or the velocity of the particle in the horizontal direction.

$y'(t) = \frac{dy}{dt}$ is the rate at which the y-coordinate is changing with respect to t or the velocity of the particle in the vertical direction.

$\langle x(t), y(t) \rangle$ is the **position vector** at any time t.
$\langle x'(t), y'(t) \rangle$ is the **velocity vector** at any time t.
$\langle x''(t), y''(t) \rangle$ is the **acceleration vector** at any time t.

$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is the **speed of the particle** or the **magnitude (length) of the velocity vector**.

$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ is the **length of the arc (or arc length) of the curve from** $t = a$ to $t = b$ or the **distance traveled by the particle from** $t = a$ to $t = b$.

Most textbooks do not contain the types of problems on vectors that are found on the AP Exam, so I supplement with the examples and worksheets below.

Example 1 (no calculator):

A particle moves in the *xy*-plane so that at any time t, the position of the particle is given by $x(t) = t^3 + 4t^2$, $y(t) = t^4 - t^3$.

(a) Find the velocity vector when $t = 1$.

Solution:

$$v(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle \frac{d}{dt}\left(t^3 + 4t^2\right), \frac{d}{dt}\left(t^4 - t^3\right) \right\rangle = \left\langle 3t^2 + 8t, 4t^3 - 3t^2 \right\rangle$$

$$v(1) = \langle 11, 1 \rangle$$

(b) Find the acceleration vector when $t = 2$.

Solution:

$$a(t) = \left\langle \frac{d}{dt}\left(\frac{dx}{dt}\right), \frac{d}{dt}\left(\frac{dy}{dt}\right) \right\rangle = \left\langle \frac{d}{dt}\left(3t^2 + 8t\right), \frac{d}{dt}\left(4t^3 - 3t^2\right) \right\rangle = \left\langle 6t + 8, 12t^2 - 6t \right\rangle$$

$$a(2) = \langle 20, 36 \rangle$$

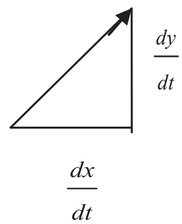
Example 2 (no calculator):

A particle moves in the *xy*-plane so that at any time t, $t \geq 0$, the position of the particle is given by $x(t) = t^2 + 3t$, $y(t) = t^3 - 3t^2$. Find the magnitude of the velocity vector when $t = 1$.

Solution:

The magnitude or length of the velocity vector can be found by using the Pythagorean Theorem, since the horizontal and vertical components make a right triangle, with the vector itself as the hypotenuse. Therefore its length is given by:

$$\text{Magnitude of velocity vector} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$



For our problem, $\frac{dx}{dt} = \frac{d}{dt}\left[t^2 + 3t\right] = 2t + 3$ and $\frac{dy}{dt} = \frac{d}{dt}\left[t^3 - 3t^2\right] = 3t^2 - 6t$.

$$\text{Magnitude of velocity vector} = \sqrt{(2t+3)^2 + \left(3t^2 - 6t\right)^2}\Bigg|_{t=1} = \sqrt{25+9} = \sqrt{34}.$$

Notice that the formula for the magnitude of the velocity vector is the same as the formula for the speed of the vector, which makes sense since speed is the magnitude of velocity.

Example 3 (no calculator):

A particle moves in the *xy*-plane so that
$x = \sqrt{3} - 4\cos t$ and $y = 1 - 2\sin t$, where $0 \le t \le 2\pi$.

The path of the particle intersects the x-axis twice. Write an expression that represents the distance traveled by the particle between the two x-intercepts. Do not evaluate.

Solution:

The path of the particle intersects the x-axis at the points where the y-component is equal to zero. Note that $1 - 2\sin t = 0$ when $\sin t = \frac{1}{2}$. For $0 \le t \le 2\pi$, this will occur when

$t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$. Since $\frac{dx}{dt} = \frac{d}{dt}\left[\sqrt{3} - 4\cos t\right] = 4\sin t$ and $\frac{dy}{dt} = \frac{d}{dt}\left[1 - 2\sin t\right] = -2\cos t$,

the distance traveled by the particle is Distance $= \int_{\pi/6}^{5\pi/6} \sqrt{(4\sin t)^2 + (-2\cos t)^2}\, dt$.

Day 4 Homework

Use your calculator on problems 10 and 13c only.

1. If $x = t^2 - 1$ and $y = e^{t^3}$, find $\dfrac{dy}{dx}$.
2. If a particle moves in the xy-plane so that at any time $t > 0$, its position vector is $\left\langle \ln\left(t^2 + 5t\right), 3t^2 \right\rangle$, find its velocity vector at time $t = 2$.
3. A particle moves in the xy-plane so that at any time t, its coordinates are given by $x = t^5 - 1$ and $y = 3t^4 - 2t^3$. Find its acceleration vector at $t = 1$.
4. If a particle moves in the xy-plane so that at time t its position vector is $\left\langle \sin\left(3t - \dfrac{\pi}{2}\right), 3t^2 \right\rangle$, find the velocity vector at time $t = \dfrac{\pi}{2}$.
5. A particle moves on the curve $y = \ln x$ so that its x-component has derivative $x'(t) = t + 1$ for $t \geq 0$. At time $t = 0$, the particle is at the point (1, 0). Find the position of the particle at time $t = 1$.
6. A particle moves in the xy-plane in such a way that its velocity vector is $\left\langle 1 + t, t^3 \right\rangle$. If the position vector at $t = 0$ is $\left\langle 5, 0 \right\rangle$, find the position of the particle at $t = 2$.
7. A particle moves along the curve $xy = 10$. If $x = 2$ and $\dfrac{dy}{dt} = 3$, what is the value of $\dfrac{dx}{dt}$?
8. The position of a particle moving in the xy-plane is given by the parametric equations $x = t^3 - \dfrac{3}{2}t^2 - 18t + 5$ and $y = t^3 - 6t^2 + 9t + 4$. For what value(s) of t is the particle at rest?
9. A curve C is defined by the parametric equations $x = t^3$ and $y = t^2 - 5t + 2$. Write the equation of the line tangent to the graph of C at the point (8, –4).
10. A particle moves in the xy-plane so that the position of the particle is given by $x(t) = 5t + 3\sin t$ and $y(t) = (8 - t)(1 - \cos t)$. Find the velocity vector at the time when the particle's horizontal position is $x = 25$.
11. The position of a particle at any time $t \geq 0$ is given by $x(t) = t^2 - 3$ and $y(t) = \dfrac{2}{3}t^3$.
    (a) Find the magnitude of the velocity vector at time $t = 5$.

(b) Find the total distance traveled by the particle from $t = 0$ to $t = 5$.

(c) Find $\dfrac{dy}{dx}$ as a function of x.

12. Point $P(x, y)$ moves in the xy-plane in such a way that $\dfrac{dx}{dt} = \dfrac{1}{t+1}$ and $\dfrac{dy}{dt} = 2t$ for $t \geq 0$.

(a) Find the coordinates of P in terms of t given that, when $t = 1$, $x = \ln 2$ and $y = 0$.

(b) Write an equation expressing y in terms of x.

(c) Find the average rate of change of y with respect to x as t varies from 0 to 4.

(d) Find the instantaneous rate of change of y with respect to x when $t = 1$.

13. Consider the curve C given by the parametric equations $x = 2 - 3\cos t$ and $y = 3 + 2\sin t$, for $-\dfrac{\pi}{2} \leq t \leq \dfrac{\pi}{2}$.

(a) Find $\dfrac{dy}{dx}$ as a function of t.

(b) Find the equation of the tangent line at the point where $t = \dfrac{\pi}{4}$.

(c) The curve C intersects the y-axis twice. Approximate the length of the curve between the two y-intercepts.

Answers to Day 4 Homework

1. $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[e^{t^3}\right]}{\frac{d}{dt}\left[t^2 - 1\right]} = \frac{3t^2 e^{t^3}}{2t} = \frac{3te^{t^3}}{2}.$$

2. $$v(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle \frac{2t+5}{t^2+5t}, 6t \right\rangle \text{ so } v(2) = \left\langle \frac{9}{14}, 12 \right\rangle.$$

3. $$v(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle 5t^4, 12t^3 - 6t^2 \right\rangle.$$

$$a(t) = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle = \left\langle 20t^3, 36t^2 - 12t \right\rangle, \text{ so } a(1) = \langle 20, 24 \rangle.$$

4. $$v(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle 3\cos\left(3t - \frac{\pi}{2}\right), 6t \right\rangle \text{ so } v\left(\frac{\pi}{2}\right) = \langle 3\cos\pi, 3\pi \rangle = \langle -3, 3\pi \rangle.$$

5. $x(t) = \int (t+1)\,dt = \frac{t^2}{2} + t + C.$

$x(0) = 1 = C$ so $x(t) = \frac{t^2}{2} + t + 1.$

Since $x(1) = \frac{5}{2}$ and $y(1) = \ln\left(\frac{5}{2}\right)$, Position $= \left(\frac{5}{2}, \ln\left(\frac{5}{2}\right)\right).$

Or, since $x(1) = 1 + \int_0^1 (t+1)\,dt = \frac{5}{2}$ and $y(1) = \ln\left(\frac{5}{2}\right)$, Position $= \left(\frac{5}{2}, \ln\left(\frac{5}{2}\right)\right).$

6. $x(t) = \int (1+t)\,dt = t + \frac{t^2}{2} + C.$ $x(0) = 5$ so $C = 5$ and $x(t) = t + \frac{t^2}{2} + 5.$

$y(t) = \int t^3\,dt = \frac{t^4}{4} + D.$ $y(0) = 0$ so $D = 0$ and $y(t) = \frac{t^4}{4}.$

Position vector $= \left\langle t + \frac{t^2}{2} + 5, \frac{t^4}{4} \right\rangle.$ At $t = 2$, Position $= (9, 4).$

Or, since $x(2) = 5 + \int_0^2 (1+t)\,dt = 9$ and $y(2) = 0 + \int_0^2 t^3\,dt = 4,$

Position $= (9, 4).$

7. When $x = 2$, $y = 5$.

$x\frac{dy}{dt} + y\frac{dx}{dt} = 0$

$(2)(3) + 5\frac{dx}{dt} = 0$ so $\frac{dx}{dt} = -\frac{6}{5}$

Or find that $\frac{dy}{dx} = -\frac{10}{x^2}$. Then substituting into $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ gives $\frac{-10}{4} = \frac{3}{\frac{dx}{dt}}$ so that $\frac{dx}{dt} = -\frac{6}{5}.$

8. $x'(t) = 3t^2 - 3t - 18 = 3(t-3)(t+2) = 0$ when $t = 3$ and $t = -2$.

$y'(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3) = 0$ when $t = 3$ and $t = 1$.

The particle is at rest when $v(t) = \langle 0, 0 \rangle$ so at rest when $t = 3$.

9. $t^3 = 8 \qquad t^2 - 5t + 2 = -4$

$t = 2 \qquad t^2 - 5t + 6 = 0$

$(t-3)(t-2) = 0$

$t = 3,\ t = 2$

At $(8, -4)$ when $t = 2$

$$\left.\frac{dy}{dx}\right|_{t=2} = \left.\frac{2t-5}{3t^2}\right|_{t=2} = -\frac{1}{12}$$

Tangent line equation: $y + 4 = -\frac{1}{12}(x - 8)$

10. $5t + 3\sin t = 25$ when $t = 5.445755...$

$$v(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \langle 5 + 3\cos t, -1 + \cos t + (8-t)\sin t \rangle$$

$$v(5.445755...) = \langle 7.008, -2.228 \rangle$$

11. (a) Magnitude when $t = 5$ is $\left.\sqrt{(2t)^2 + (2t^2)^2}\right|_{t=5} = \sqrt{2600}$ or $10\sqrt{26}$

(b)

$$\text{Distance } = \int_0^5 \sqrt{(2t)^2 + (2t^2)^2}\,dt = \int_0^5 2t\sqrt{1+t^2}\,dt = \left.\frac{2}{3}(1+t^2)^{\frac{3}{2}}\right|_0^5 = \frac{2}{3}\left(26^{\frac{3}{2}} - 1\right)$$

(c) $\frac{dy}{dx} = \frac{2t^2}{2t} = t = \sqrt{x+3}$

12. (a) $x(t)=\int \frac{1}{t+1}\,dt=\ln(t+1)+C.$ When $t=1,\ x=\ln 2$ so $C=0.$

$x(t)=\ln(t+1)$

$y(t)=\int 2t\,dt=t^2+D.$ When $t=1, y=0$ so $D=-1.$

$y(t)=t^2-1$

$(x,y)=\left(\ln(t+1),\ t^2-1\right)$

(b) $t+1=e^x$ so $t=e^x-1$ and $y=\left(e^x-1\right)^2-1=e^{2x}-2e^x.$

(c) Average rate of change $=\dfrac{y(b)-y(a)}{x(b)-x(a)}=\dfrac{y(4)-y(0)}{x(4)-x(0)}=\dfrac{15-(-1)}{\ln 5-\ln 1}=\dfrac{16}{\ln 5}$

(d) Instantaneous rate of change $=\left.\dfrac{dy}{dx}\right|_{t=1}=\left.\dfrac{2t}{\frac{1}{t+1}}\right|_{t=1}=4$

13. (a) $\dfrac{dy}{dx}=\dfrac{2\cos t}{3\sin t}=\dfrac{2}{3}\cot t$

(b) $y-\left(3+\sqrt{2}\right)=\dfrac{2}{3}\left(x-\left(2-\dfrac{3\sqrt{2}}{2}\right)\right)$

(c) $x=0$ when $t=-0.84106867\ldots,\ 0.84106867\ldots$

length $=\int_{-0.841\ldots}^{0.841,,,}\sqrt{(3\sin t)^2+(2\cos t)^2}\,dt=3.756$ or 3.757

Day 5: Motion Along a Curve — Vectors (continued)

Example (calculator):

An object moving along a curve in the xy-plane has position $\left(x(t), y(t)\right)$ at time t with $\dfrac{dx}{dt} = \sin\left(t^3\right)$, $\dfrac{dy}{dt} = \cos\left(t^2\right)$. At time $t = 2$, the object is at the position (1, 4).

(a) Find the acceleration vector for the particle at $t = 2$.

(b) Write the equation of the tangent line to the curve at the point where $t = 2$.

(c) Find the speed of the vector at $t = 2$.

(d) Find the position of the particle at time $t = 1$.

Solution:

(a) Students should use their calculators to numerically differentiate both $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ when $t = 2$ to get $a(2) = \langle -1.476, 3.027 \rangle$.

(b) When $t = 2$, $\dfrac{dy}{dx} = \dfrac{\cos 4}{\sin 8}$ or -0.661, so the tangent line equation is

$y - 4 = \dfrac{\cos 4}{\sin 8}(x - 1)$ or $y - 4 = -0.661(x - 1)$.

Notice that it is fine to leave the slope as the exact value or to write it as a decimal correct to three decimal places.

(c) Speed $= \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2} = \sqrt{(\sin 8)^2 + (\cos 4)^2}$ or 1.186

Notice that it is fine to leave the speed as the exact value or to write it as a decimal correct to three decimal places.

(d) Students should apply the Fundamental Theorem of Calculus to find the x and y components of the position.

$$x(1) = x(2) - \int_1^2 x'(t)\,dt \qquad y(1) = y(2) - \int_1^2 y'(t)\,dt$$

$$= 1 - \int_1^2 \sin\left(t^3\right)dt \qquad = 4 - \int_1^2 \cos\left(t^2\right)dt$$

$$= 0.782 \qquad = 4.443$$

Therefore the position at time $t = 1$ is (0.782, 4.443).

Day 5 Homework

Use your calculator on problems 7–11 only.

1. If $x = e^{2t}$ and $y = \sin(3t)$, find $\dfrac{dy}{dx}$ in terms of t.

2. Write an integral expression to represent the length of the path described by the parametric equations $x = \cos^3 t$ and $y = \sin^2 t$ for $0 \le t \le \dfrac{\pi}{2}$.

3. For what value(s) of t does the curve given by the parametric equations $x = t^3 - t^2 - 1$ and $y = t^4 + 2t^2 - 8t$ have a vertical tangent?

4. For any time $t \ge 0$, if the position of a particle in the xy-plane is given by $x = t^2 + 1$ and $y = \ln(2t + 3)$, find the acceleration vector.

5. Find the equation of the tangent line to the curve given by the parametric equations $x(t) = 3t^2 - 4t + 2$ and $y(t) = t^3 - 4t$ at the point on the curve where $t = 1$.

6. If $x(t) = e^t + 1$ and $y = 2e^{2t}$ are the equations of the path of a particle moving in the xy-plane, write an equation for the path of the particle in terms of x and y.

7. A particle moves in the xy-plane so that its position at any time t is given by $x = \cos(5t)$ and $y = t^3$. What is the speed of the particle when $t = 2$?

8. The position of a particle at time $t \ge 0$ is given by the parametric equations $x(t) = \dfrac{(t-2)^3}{3} + 4$ and $y(t) = t^2 - 4t + 4$.

(a) Find the magnitude of the velocity vector at $t = 1$.

(b) Find the total distance traveled by the particle from $t = 0$ to $t = 1$.

(c) When is the particle at rest? What is its position at that time?

9. An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \ge 0$ with $\dfrac{dx}{dt} = 1 + \tan(t^2)$ and $\dfrac{dy}{dt} = 3e^{\sqrt{t}}$. Find the acceleration vector and the speed of the object when $t = 5$.

10. A particle moves in the xy-plane so that the position of the particle is given by $x(t) = t + \cos t$ and $y(t) = 3t + 2\sin t$, $0 \le t \le \pi$. Find the velocity vector when the particle's vertical position is $y = 5$.

11. An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with $\frac{dx}{dt} = 2\sin(t^3)$ and $\frac{dy}{dt} = \cos(t^2)$ for $0 \le t \le 4$. At time $t = 1$, the object is at the position (3, 4).

(a) Write an equation for the line tangent to the curve at (3, 4).

(b) Find the speed of the object at time $t = 2$.

(c) Find the total distance traveled by the object over the time interval $0 \le t \le 1$.

(d) Find the position of the object at time $t = 2$.

Answers to Day 5 Homework

1. $\frac{dy}{dx} = \frac{3\cos(3t)}{2e^{2t}}$

2. Length = $\int_0^{\frac{\pi}{2}} \sqrt{9\cos^4 t \sin^2 t + 4\sin^2 t \cos^2 t}\, dt$

3. $\frac{dy}{dx} = \frac{4t^3 + 4t - 8}{3t^2 - 2t}$ is undefined when $3t^2 - 2t = 0$.

So the curve given by the parametric equations $x = t^3 - t^2 - 1$ and $y = t^4 + 2t^2 - 8t$ has a vertical tangent when $t = 0$ and $t = \frac{2}{3}$.

4. $v(t) = \left\langle 2t, \frac{2}{2t+3} \right\rangle$, $a(t) = \left(2, -\frac{4}{(2t+3)^2} \right)$

5. $\left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{3t^2 - 4}{6t - 1}\right|_{t=1} = -\frac{1}{2}$. When $t = 1$, $x = 1$, $y = -3$.

Tangent line equation: $y + 3 = -\frac{1}{2}(x - 1)$

6. $e^t = x - 1$ so $e^{2t} = x^2 - 2x + 1$. Then $y = 2e^{2t}$ so $y = 2x^2 - 4x + 2$.

7. Speed = $\left.\sqrt{\left(-5\sin(5t)\right)^2 + \left(3t^2\right)^2}\right|_{t=2} = 12.304$

8. **(a)** Magnitude = $\left.\sqrt{(t-2)^4+(2t-4)^2}\right|_{t=1}=\sqrt{5}$

(b) Distance = $\int_0^1 \sqrt{(t-2)^4+(2t-4)^2}\,dt$ = 3.816

(c) The particle is at rest when $v(t)=\left\langle (t-2)^2,\ 2t-4\right\rangle=\langle 0,0\rangle$, so is at rest when $t = 2$. Position = (4, 0)

9. $a(5)=\langle 10.178,\ 6.277\rangle$, speed = $\left.\sqrt{\left(1+\tan\left(t^2\right)\right)^2+\left(3e^{\sqrt{t}}\right)^2}\right|_{t=5}$ = 28.083

10. $3t+2\sin t=5$ when $t=1.079...$

$v(1.079...)=\langle 0.119,\ 3.944\rangle$

11. (a) $\dfrac{dy}{dx}=\left.\dfrac{\cos\left(t^2\right)}{2\sin\left(t^3\right)}\right|_{t=1}=0.321$

Tangent line equation: $y-4=0.321(x-3)$

(b) Speed = $\left.\sqrt{4\sin^2\left(t^3\right)+\cos^2\left(t^2\right)}\right|_{t=2}=2.084$

(c) Distance = $\int_0^1\sqrt{4\sin^2\left(t^3\right)+\cos^2\left(t^2\right)}\,dt=1.126$

(d) $x(2)=3+\int_1^2 2\sin\left(t^3\right)dt=3.436,\ \ y(2)=4+\int_1^2\cos\left(t^2\right)dt=3.557$ so position

= (3.436, 3.557)

Day 6: Motion Along a Curve — Vectors (continued)

I don't work any examples on Day 6. The students usually need a little more practice on vectors, but no new material is covered in the Day 6 homework.

Day 6 Homework

Use your calculator only on problems 3–7.

1. The position of a particle at any time $t \geq 0$ is given by $x(t) = t^2 - 2,\ y(t) = \frac{2}{3}t^3$.

(a) Find the magnitude of the velocity vector at $t = 2$.

(b) Set up an integral expression to find the total distance traveled by the particle from $t = 0$ to $t = 4$.

(c) Find $\frac{dy}{dx}$ as a function of x.

(d) At what time t is the particle on the y-axis? Find the acceleration vector at this time.

2. An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with the velocity vector $v(t) = \left(\frac{1}{t+1}, 2t\right)$. At time $t = 1$, the object is at (ln 2, 4).

(a) Find the position vector.

(b) Write an equation for the line tangent to the curve when $t = 1$.

(c) Find the magnitude of the velocity vector when $t = 1$.

(d) At what time $t > 0$ does the line tangent to the particle at $(x(t), y(t))$ have a slope of 12?

3. A particle moving along a curve in the xy-plane has position $(x(t), y(t))$, with $x(t) = 2t + 3\sin t$ and $y(t) = t^2 + 2\cos t$, where $0 \leq t \leq 10$. Find the velocity vector at the time when the particle's vertical position is $y = 7$.

4. A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with $\frac{dx}{dt} = 1 + \sin(t^3)$. The derivative $\frac{dy}{dt}$ is not explicitly given. For any time t, $t \geq 0$, the line tangent to the curve at $(x(t), y(t))$ has a slope of $t + 3$. Find the acceleration vector of the object at time $t = 2$.

5. An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with $\frac{dx}{dt} = \cos(e^t)$ and $\frac{dy}{dt} = \sin(e^t)$ for $0 \le t \le 2$. At time $t = 1$, the object is at the point (3, 2).

(a) Find the equation of the tangent line to the curve at the point where $t = 1$.

(b) Find the speed of the object at $t = 1$.

(c) Find the total distance traveled by the object over the time interval $0 \le t \le 2$.

(d) Find the position of the object at time $t = 2$.

6. A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with $\frac{dx}{dt} = \sin(t^3 - t)$ and $\frac{dy}{dt} = \cos(t^3 - t)$. At time $t = 3$, the particle is at the point (1, 4).

(a) Find the acceleration vector for the particle at $t = 3$.

(b) Find the equation of the tangent line to the curve at the point where $t = 3$.

(c) Find the magnitude of the velocity vector at $t = 3$.

(d) Find the position of the particle at time $t = 2$.

7. An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with $\frac{dy}{dt} = 2 + \sin(e^t)$. The derivative $\frac{dx}{dt}$ is not explicitly given. At $t = 3$, the object is at the point (4, 5).

(a) Find the y-coordinate of the position at time $t = 1$.

(b) At time $t = 3$, the value of $\frac{dy}{dx}$ is -1.8. Find the value of $\frac{dx}{dt}$ when $t = 3$.

(c) Find the speed of the object at time $t = 3$.

Answers to Day 6 Homework

1. **(a)** Magnitude $= \left.\sqrt{(2t)^2 + (2t^2)^2}\right|_{t=2} = 4\sqrt{5}$

(b) Distance $= \int_0^4 \sqrt{4t^2 + 4t^4}\, dt$

(c) $\dfrac{dy}{dx} = \dfrac{2t^2}{2t} = t = \sqrt{x+2}$

(d) Particle is on the y-axis when $t = \sqrt{2}$, and $\dfrac{dx}{dt} = 2t - 1$

2. **(a)** $x(t) = \int \dfrac{1}{t+1}\, dt = \ln(t+1) + C$. Since $x(1) = \ln 2$, $C = 0$.

$y(1) = 4,\ D = 3$. Since $y(1) = 4,\ D = 3$.

Position vector $= \left(\ln(t+1), t^2 + 3\right)$

(b) When $t = 1$, $\dfrac{dy}{dx} = \dfrac{2}{\frac{1}{2}} = 4$ so the tangent line equation is $y - 4 = 4(x - \ln 2)$.

(c) Magnitude $= \left.\sqrt{\left(\dfrac{1}{t+1}\right)^2 + (2t)^2}\right|_{t=1} = \dfrac{\sqrt{17}}{2}$

(d) $\dfrac{dy}{dx} = \dfrac{2t}{\frac{1}{t+1}} = 2t(t+1) = 12$ when $2t^2 + 2t - 12 = 0$ so $t = 2$

3. $t^2 + 2\cos t = 7$ when $t = 2.996...$ $\quad v(2.996...) = \langle -0.968,\ 5.704 \rangle$

4. $\dfrac{dy}{dt} = \left(\dfrac{dx}{dt}\right)(t+3) = \left(1 + \sin t^3\right)(t+3)$ so $a(2) = \langle -1.746,\ -6.741 \rangle$

5. **(a)** When $t = 1$,

$\dfrac{dy}{dx} = \left.\dfrac{\sin(e^t)}{\cos(e^t)}\right|_{t=1} = -0.451$ so the tangent line equation is $y - 2 = -0.451(x - 3)$

(b) Speed $= \left.\sqrt{\left(\cos(e^t)\right)^2 + \left(\sin(e^t)\right)^2}\right|_{t=1} = 1$

(c) Distance $= \int_0^2 \sqrt{\left(\cos(e^t)\right)^2 + \left(\sin(e^t)\right)^2}\, dt = 2$

(d) $x(2) = 3 + \int_1^2 \cos(e^t)\, dt = 2.896,\ y(2) = 2 + \int_1^2 \sin(e^t)\, dt = 1.676$

so position = (2.896, 1.676)

6. **(a)** $a(3) = \langle 11.029,\ 23.545 \rangle$

(b)

$\dfrac{dy}{dx} = \left.\dfrac{\cos(t^3 - t)}{\sin(t^3 - t)}\right|_{t=3} = -0.468$ so the tangent line equation is $y - 4 = -0.468(x - 1)$

(c) Magnitude $= \left.\sqrt{\left(\sin(t^3 - t)\right)^2 + \left(\cos(t^3 - t)\right)^2}\right|_{t=3} = 1$

(d) $x(2) = 1 - \int_2^3 \sin(t^3 - t)\,dt = 0.932,\ y(2) = 4 - \int_2^3 \cos(t^3 - t)\,dt = 4.002$ so the position = (0.932, 4.002)

7. **(a)** $y(1) = 5 - \int_1^3 \left(2 + \sin(e^t)\right)dt = 1.269$

(b) $\dfrac{dy}{dx} = \dfrac{2 + \sin(e^t)}{\frac{dx}{dt}} = -1.8$ so $\left.\dfrac{dx}{dt}\right|_{t=3} = \dfrac{2 + \sin(e^3)}{-1.8} = -1.636$

(c) Speed $= \sqrt{\left(\dfrac{2 + \sin(e^3)}{-1.8}\right)^2 + \left(2 + \sin(e^3)\right)^2} = 3.368$

About the Author

Nancy Stephenson teaches at Clements High School in Sugar Land, Texas. She was a member of the AP Calculus Development Committee from 1999 to 2003 and is a College Board consultant.