

# AP<sup>®</sup> Calculus

2007–2008  
Professional Development  
Workshop Materials

**Special Focus:  
Approximation**

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## Introduction and Background

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“All exact science is dominated by the idea of approximation.” - Bertrand Russell

The origin of mathematics is probably rooted in the practical need to count (Eves, Burton). For example, there is some evidence to suggest that very early peoples may have kept track of the number of days since the last full moon. Distinct sounds may have been used initially, eventually leading to the use of tally marks or notches. It soon became necessary to use other, more permanent symbols as representations of tallies. Number systems were created.

There is evidence the ancient Babylonians, Egyptians, Chinese, and Greeks all worked on mathematical problems. For example, around 1950 BCE the Babylonians were able to solve some quadratic equations, and about 440 BCE Hippocrates (100 years before Euclid) wrote about geometry in his *Elements* (Burton, pp. 118–119).

As the science of mathematics grew, the problem of approximation became an increasing challenge. The discovery of irrational numbers and transcendental functions led to the need for approximation (Steffens). Several cultures found a numerical approximation of  $\pi$ . The Babylonians used  $25/8$ , the Chinese used 3.141014 (in CE 263), and in the Middle Ages a Persian computed  $\pi$  to 16 digits. Euler, Laplace, Fourier, and Chebyshev each contributed important works involving approximation.

Despite the prevalence and importance of approximation throughout the history of mathematics, very few approximation questions were asked on the AP Calculus Exam until the introduction of graphing calculators in 1994–95. There were occasional questions concerning a tangent line approximation, a Riemann sum, or an error estimate in a series approximation prior to 1995. However, since students were without graphing calculators, even these few problems had to result in nice, round numbers.

Calculus reform, the emphasis on conceptual understanding, the desire to solve more real-world problems, and powerful graphing calculators now allow us to (teach and) ask more challenging, practical approximation problems. Most (more than two-thirds of) AP Calculus Exam approximation problems have appeared since 1997. Tangent line (or local linear) approximation problems, definite integral approximations, and error estimates using series appeared before 1995. Questions involving an approximation to a derivative and Euler’s method began in 1998.

Below, Larry Riddle has provided a fine summary of the approximation problems on both the multiple-choice and free-response sections of the AP Calculus Exam. He also provides

some of the more recent exam questions in order to illustrate how approximation concepts have been tested. This summary table and example problem set is an excellent place to start in order to prepare your students for the type of approximation problems that might appear on the AP Calculus Exam.

There are several expository articles, each focused on a specific approximation topic. These notes provide essential background material in order to understand and successfully teach each concept. In addition, there are four classroom explorations and two instructional units. The latter are complete lessons concerning specific approximation topics used by experienced AP Calculus teachers. Teachers should carefully consider the material in this Special Focus section and pick and choose from it, reorganize it, and build upon it to create classroom experiences that meet the needs of their students.

It is our hope that these articles will help teachers and students better prepare for approximation problems on the AP Calculus Exam. Although there is no way to predict what type of approximation problem will appear on the next AP Exam, a review of previous questions has always been effective. The AP Calculus community is extremely supportive and we believe this material will help our students better understand approximation concepts and succeed on the exam.

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## Using Approximations in a Variety of AP Questions

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Approximation techniques involving derivatives, integrals, and Taylor polynomials have been tested on the AP Calculus Exams from the very beginning. With the transition to the use of graphing calculators and the changes to the AP Calculus Course Description in the mid-1990s, however, the emphasis on approximations became a more fundamental component of the course. The following table lists various approximation problems from the free-response sections and the released multiple-choice sections, arranged according to themes listed in the topic outlines for Calculus AB and Calculus BC in the AP Calculus Course Description. More than two-thirds of the problems have appeared since 1997.

### Tangent Line Approximation (Local Linear Approximation)

Free Response	Multiple Choice
1991 AB3	1969 AB/BC 36
1995 AB3	1973 AB 44
1998 AB4	1997 AB 14
1999 BC6	1998 BC 92
2002 AB6 (over/under estimate?)	
2005 AB6	

### Approximating a Derivative Value

Free Response	Multiple Choice
1998 AB3 (at point in table or from graph)	
2001 AB2/BC2 (at point in table)	
2003 AB3 (at point not in table)	
2005 AB3/BC3 (at point not in table)	

## Approximating a Definite Integral

Free Response	Multiple Choice
1994 AB6 (trapezoid from function)	1973 AB/BC 42 (trapezoid from function)
1996 AB3/BC3 (trapezoid from function)	1988 BC 18 (trapezoid from function)
1998 AB3 (midpoint from table)	1993 AB 36 (trapezoid, left from function)
1999 AB3/BC3 (midpoint from table)	1993 BC 40 (Simpson's rule from function)
2001 AB2/BC2 (trapezoid from table)	1997 AB 89 (trapezoid from table)
2002(B) AB4/BC4 (trapezoid from graph)	1998 AB/BC 9 (estimate from graph)
2003 AB3 (left sum from table, unequal widths, over/under estimate?)	1998 AB/BC 85 (trapezoid from table, unequal widths)
2003(B) AB3/BC3 (midpoint from table)	1998 BC 91 (left from table)
2004(B) AB3/BC3 (midpoint from table)	2003 AB/BC 85 (trapezoid, right sum from graph, over/under estimate?)
2005 AB3/BC3 (trapezoid from table, unequal widths)	2003 BC 25 (right sum from table, unequal widths)
2006 AB4/BC4 (midpoint from table)	
2006(B) AB6 (trapezoid from table, unequal widths)	

## Error Estimates Using Series

Free Response	Multiple Choice
1971 BC4 (alternating series or Lagrange EB)	
1976 BC7 (Lagrange EB)	
1979 BC4 (alternating series or Lagrange EB)	
1982 BC5 (alternating series or Lagrange EB)	
1984 BC4 (alternating series)	
1990 BC5 (alternating series or Lagrange EB)	
1994 BC5 (alternating series)	
1999 BC4 (Lagrange EB)	
2000 BC3 (alternating series)	
2003 BC6 (alternating series)	
2004 BC6 (Lagrange EB)	
2004(B) BC2 (Lagrange EB)	
2006(B) BC6 (alternating series)	



## Euler's Method for Differential Equations

Free Response	Multiple Choice
1998 BC4 1999 BC6 (over/under estimate?) 2001 BC5 2002 BC5 2005 BC4 (over/under estimate?) 2006 BC5	2003 BC 5

(The scarcity of multiple-choice problems means only that those topics did not appear on an AP Released Exam. Questions from all of these approximation topics have certainly appeared in multiple-choice sections since 1997.)

Approximation techniques may not always yield “nice” answers. With the introduction of calculators on the AP Calculus Exam, some line had to be drawn in evaluating the accuracy of numerical answers reported in decimal form. The three decimal place standard has been used every year since 1995. The choice of reporting answers to three decimal places was really just a compromise; one decimal place would be too few and five decimal places would probably be too many. Note that the standard can be overridden in a specific problem. For example, in an application problem the student could be asked for an answer rounded to the nearest whole number. The Reading leadership has developed grading procedures to minimize the number of points that a student might lose for presentation errors in numerical answers.

The following problems are taken from recent AP Calculus Exams and illustrate how approximation concepts have been tested. Brief solutions are provided in the Appendix. The complete problems and the Scoring Guidelines are available at AP Central®.

## 2006 AB4/BC4

$t$ (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

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Rocket *A* has positive velocity  $v(t)$  after being launched upward from an initial height of 0 feet at time  $t = 0$  seconds. The velocity of the rocket is recorded for selected values of  $t$  over the interval  $0 \leq t \leq 80$  seconds, as shown in the table above.

- (b) Using correct units, explain the meaning of  $\int_{10}^{70} v(t) dt$  in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate  $\int_{10}^{70} v(t) dt$

**Comments:**

Students needed to pick out the correct intervals and the midpoints of those intervals to use for the midpoint Riemann sum approximation since the limits on the definite integral did not include the full range of the data given in the table. Equally important in this problem was the knowledge about what the approximation represented, including the correct units, not just the ability to do the computation.

A plot of the data suggests that the graph of  $v(t)$  is concave down. This is also suggested by the difference quotients between successive data points (since they are decreasing). Assuming that  $v(t)$  is concave down, another natural question that could have been asked would have been whether the midpoint approximation overestimates or underestimates the actual value of the definite integral.

**2005 AB3/BC3**

Distance $x$ (cm)	0	1	5	6	8
Temperature $T(x)$ ( $^{\circ}\text{C}$ )	100	93	70	62	55

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature  $T(x)$  in degrees Celsius ( $^{\circ}\text{C}$ ), of the wire  $x$  cm from the heated end. The function  $T$  is decreasing and twice differentiable.

- (a) Estimate  $T'(7)$ . Show the work that leads to your answer. Indicate units of measure.
- (b) Write an integral expression in terms of  $T(x)$  for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

**Comments:**

In part (a) the students were asked to approximate the value of the derivative at a point that is not in the table. Students were expected to find the “best” estimate for the derivative at  $x = 7$  by using a symmetric difference quotient with  $x = 6$  and  $x = 8$ .

The subintervals in the table are not of equal length. The trapezoid rule can therefore not be used. Students needed to either add the areas of the four individual trapezoids, or average the left and right Riemann sums. In these types of problems it is also important for students to show the setup for the computations, particularly on the calculator section of the exam.

### 2005 BC4

Consider the differential equation  $\frac{dy}{dx} = 2x - y$ .

- (c) Let  $y = f(x)$  be the particular solution to the given differential equation with the initial condition  $f(0) = 1$ . Use Euler's method, starting at  $x = 0$  with two steps of equal size, to approximate  $f(-0.4)$ . Show the work that leads to your answer.
- (d) Find  $\frac{d^2y}{dx^2}$  in terms of  $x$  and  $y$ . Determine whether the approximation found in part (c) is less than or greater than  $f(-0.4)$ . Explain your reasoning.

#### Comments:

Notice that Euler's method is going in "backwards" steps, so  $\Delta x = -0.2$ . Students need experience with doing the computations for both directions.

The determination of whether the approximation is less than or greater than the actual value is based on the sign of the second derivative over an interval, not just at the starting point.

Here  $\frac{d^2y}{dx^2} = 2 - 2x + y$ . This is positive for  $x \leq 0$  and  $y \geq 0$ . Thus the tangent lines will lie underneath the graphs of solution curves in this quadrant. For more discussion about this problem, see Steve Greenfield's article "Don't Forget the Differential Equation: Finishing 2005 BC4," which can be found under the Teaching Resource Materials on the AP Calculus AB and BC Exam home pages at AP Central.

### 2004 (Form B) BC2

Let  $f$  be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for  $f$  about  $x = 2$  is given by  $T(x) = 7 - 9(x - 2)^2 - 3(x - 2)^3$ .

- (c) Use  $T(x)$  to find an approximation for  $f(0)$ . Is there enough information given to determine whether  $f$  has a critical point at  $x = 0$ ? If not, explain why not. If so, determine whether  $f(0)$  is a relative maximum, a relative minimum, or neither, and justify your answer.
- (d) The fourth derivative of  $f$  satisfies the inequality  $|f^{(4)}(x)| \leq 6$  for all  $x$  in the closed interval  $[0, 2]$ . Use the Lagrange error bound on the approximation to  $f(0)$  found in part (c) to explain why  $f(0)$  is negative.

#### Comments:

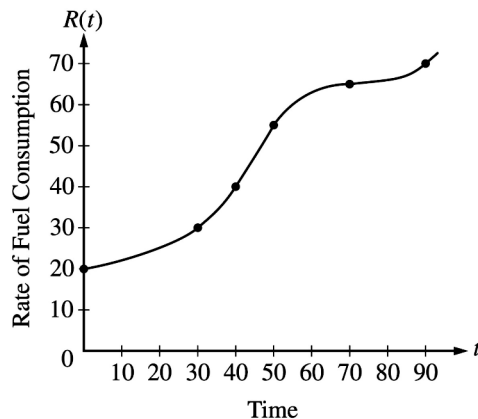
The Taylor polynomial approximates the function near  $x = 2$ . However,  $T(x)$  gives exact information only at  $x = 2$ .

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The Lagrange error bound in part (d) is  $\max_{0 \leq x \leq 2} |f^{(4)}(x)| \frac{(2-0)^4}{4!} \leq 6 \cdot \frac{16}{24} = 4$ .

So  $|f(0) - T(0)| \leq 4$  and therefore  $f(0) \leq T(0) + 4 = -1 < 0$ .

**2003 AB3**



$t$ (minutes)	$R(t)$ (gallons per minute)
0	20
30	30
40	40
50	55
70	65
90	70

The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice-differentiable and strictly increasing function  $R$  of time  $t$ . The graph of  $R$  and a table of selected values of  $R(t)$  for the time interval  $0 \leq t \leq 90$  minutes are shown above.

- (a) Use data from the table to find an approximation for  $R'(45)$ . Show the computations that led to your answer. Indicate units of measure.
- (c) Approximate the value of  $\int_0^{90} R(t) dt$  using a left Riemann sum with the five subintervals indicated by the data in the table. Is this numerical approximation less than the value of  $\int_0^{90} R(t) dt$ ? Explain your reasoning.

**Comments:**

Here is another approximation of a derivative at a point not in the table. A symmetric difference quotient based on  $t = 40$  and  $t = 50$  gives the best estimate. This is also supported by the graph, which shows that the slope of the secant line between 40 and 50 is a good approximation to the tangent line at  $t = 45$ .

The subintervals in the table are of unequal lengths, so care must be taken when computing the areas of each left rectangle. It is also important for students to know whether a left or right Riemann sum is too large or too small when the graph of the function is increasing or decreasing.

**2003 BC6**

The function  $f$  is defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots + \frac{(-1)^n x^{2n}}{(2n+1)!} + \dots$$

for all real numbers  $x$ .

- (b) Show that  $1 - \frac{1}{3!}$  approximates  $f(1)$  with error less than  $\frac{1}{100}$ .

**Comments:**

The series with  $x = 1$  is an alternating series whose terms are decreasing in absolute value with limit 0. Thus, the error is less than the first omitted term,  $1/5! = 1/120$ . It is important for students to recognize that all three conditions on the terms of the series are needed in order to use the usual error bound for alternating series. It is *not* sufficient to simply state that the series is alternating.

**2002 AB6**

$x$	-1.5	-1.0	-0.5	0	0.5	1.0	1.5
$f(x)$	-1	-4	-6	-7	-6	-4	-1
$f'(x)$	-7	-5	-3	0	3	5	7

Let  $f$  be a function that is differentiable for all real numbers. The table above gives the values of  $f$  and its derivative  $f'$  for selected points  $x$  in the closed interval  $-1.5 \leq x \leq 1.5$ . The second derivative of  $f$  has the property that  $f''(x) > 0$  for  $-1.5 \leq x \leq 1.5$ .

- (b) Write an equation of the line tangent to the graph of  $f$  at the point where  $x = 1$ . Use this line to approximate the value of  $f(1.2)$ . Is this approximation greater than or less than the actual value of  $f(1.2)$ ? Give a reason for your answer.

**Comments:**

The approximation is less than the actual value because the graph of  $f$  is concave up over the *entire* interval from  $x = 1$  to  $x = 1.2$ . It is not sufficient to check concavity just at  $x = 1$ . This property is due to the fact that

$$f(b) - (f(a) + f'(a)(x - a)) = \frac{1}{2}f''(c)(b - a)^2 \text{ for some } c \text{ between } a \text{ and } b$$

or, geometrically, that the tangent line lies below the graph when the graph is concave up.

**1997 AB 14 (Multiple Choice)**

Let  $f$  be a differentiable function such that  $f(3) = 2$  and  $f'(3) = 5$ . If the tangent line to the graph of  $f$  at  $x = 3$  is used to find an approximation to a zero of  $f$ , that approximation is

- (A) 0.4      (B) 0.5      (C) 2.6      (D) 3.4      (E) 5.5

**Comments:**

Newton's method is no longer part of the AP Course Description. This simple application of the tangent line approximation, however, is the basic idea of that method. The tangent line approximation is  $L(x) = 5(x - 3) + 2$ , and so  $L(x) = 0$  when  $x = 2.6$ . The answer is (C).

**1998 BC 92 (Multiple Choice)**

Let  $f$  be the function given by  $f(x) = x^2 - 2x + 3$ . The tangent line to the graph of  $f$  at  $x = 2$  is used to approximate values of  $f(x)$ . Which of the following is the greatest value of  $x$  for which the error resulting from this tangent line approximation is less than 0.5?

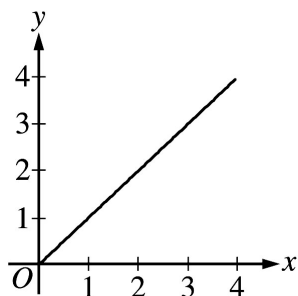
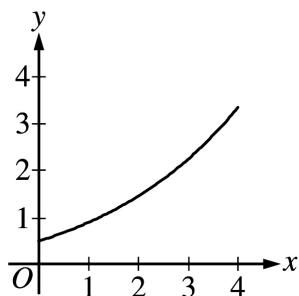
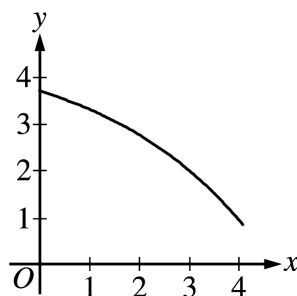
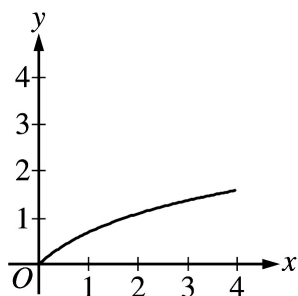
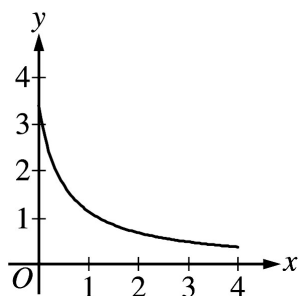
- (A) 2.4      (B) 2.5      (C) 2.6      (D) 2.7      (E) 2.8

**Comments:**

$f'(x) = 2x - 2$ ,  $f'(2) = 2$ , and  $f(2) = 3$ , so an equation for the tangent line is  $y = 2x - 1$ . The difference between the function and the tangent line is represented by  $(x - 2)^2$ . Solve  $(x - 2)^2 < 0.5$ . This inequality is satisfied for all  $x$  such that  $2 - \sqrt{0.5} < x < 2 + \sqrt{0.5}$ , or  $1.293 < x < 2.707$ . Thus, the largest value in the list that satisfies the inequality is 2.7, and the correct answer is (D). One could also investigate this problem graphically by graphing the function that is the difference between the parabola and the tangent line.

**2003 AB/BC 85 (Multiple Choice)**

If a trapezoidal sum over approximates  $\int_0^4 f(x)dx$ , and a right Riemann sum under approximates  $\int_0^4 f(x)dx$ , which of the following could be the graph of  $y = f(x)$ ?



**Comments:**

If the graph is decreasing, then  $\text{Right}(n) < \int_a^b f(x)dx < \text{Left}(n)$  for the right Riemann and left Riemann sums using  $n$  subintervals.

If the graph is concave up, then  $\text{Mid}(n) < \int_a^b f(x)dx < \text{Trap}(n)$  for the trapezoid sum and the midpoint Riemann sum using  $n$  subintervals.

Graph (A) is decreasing and concave up, and therefore could be the graph of  $y = f(x)$ .

If the graph is increasing or concave down, the respective inequalities are reversed.

**Appendix:****Free-Response Solutions**

For more details, see the Scoring Guidelines for the AB Exams and the BC Exams at AP Central.

**2006 AB4/BC4**

Since the velocity is positive,  $\int_{10}^{70} v(t)dt$  represents the distance, in feet, traveled by rocket A from  $t = 10$  seconds to  $t = 70$  seconds. A midpoint Riemann sum is  $20[v(20) + v(40) + v(60)] = 20[22 + 35 + 44] = 2020$  ft.

**2005 AB3/BC3**

$$(a) \quad \frac{T(8) - T(6)}{8 - 6} = \frac{55 - 62}{2} = -\frac{7}{2} \text{ } ^\circ\text{C/cm}$$

$$(b) \quad \frac{1}{8} \int_0^8 T(x)dx$$

Trapezoidal approximation for  $\int_0^8 T(x)dx$ :

$$A = \frac{100 + 93}{2} \cdot 1 + \frac{93 + 70}{2} \cdot 4 + \frac{70 + 62}{2} \cdot 1 + \frac{62 + 55}{2} \cdot 2$$

$$\text{Average temperature} \approx \frac{1}{8} A = 75.6875^\circ\text{C}$$

**2005 BC4**

$$(c) \quad f(-0.2) \approx f(0) + f'(0)(-0.2) = 1 + (-1)(-0.2) = 1.2$$

$$f(-0.4) \approx f(-0.2) + f'(-0.2)(-0.2) \approx 1.2 + (-1.6)(-0.2) = 1.52$$

$$(d) \quad \frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - 2x + y$$

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$\frac{d^2y}{dx^2}$  is positive in quadrant II because  $x < 0$  and  $y > 0$ . Therefore  $1.52 < f(-0.4)$  since all solution curves in quadrant II are concave up.

**2004 (Form B) BC2**

(c)  $f(0) \approx T(0) = -5$

It is not possible to determine if  $f$  has a critical point at  $x = 0$  because  $T(x)$  gives exact information only at  $x = 2$ .

(d) The Lagrange error bound is  $\max_{0 \leq x \leq 2} |f^{(4)}(x)| \frac{(2-0)^4}{4!} \leq 6 \cdot \frac{16}{24} = 4$ . So  $|f(0) - T(0)| \leq 4$  and therefore  $f(0) \leq T(0) + 4 = -1 < 0$ .

**2003 AB3**

(a)  $R'(45) \approx \frac{R(50) - R(40)}{50 - 40} = \frac{55 - 40}{10} = 1.5 \text{ gal/min}^2$

(c)  $\int_0^{90} R(t) dt \approx (30)(20) + (10)(30) + (10)(40) + (20)(55) + (20)(65) = 3700$

Yes, this approximation is less because the graph of  $R$  is increasing on the interval.

**2003 BC6**

(b)  $f(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \dots + \frac{(-1)^n}{(2n+1)!} + \dots$

This is an alternating series whose terms decrease in absolute value with limit 0. Thus, the error is less than the first omitted term, so

$$\left| f(1) - \left( 1 - \frac{1}{3!} \right) \right| \leq \frac{1}{5!} = \frac{1}{120} < \frac{1}{100}.$$

**2002 AB6**

(b)  $y = 5(x - 1) - 4$

$$f(1.2) \approx 5(0.2) - 4 = -3$$

The approximation is less than  $f(1.2)$  because the graph of  $f$  is concave up on the interval  $1 < x < 1.2$ .



## Approximating Derivative Values

Monique Morton  
Woodrow Wilson Senior High School  
Washington, D.C.

A numerical approximation to the derivative of a function at a point can be obtained in two ways: geometrically using the slope of the tangent line or by using a difference quotient. On the AP Calculus Examination in recent years (e.g., 2005, 2003), students have been expected to approximate the derivative at a point given a table of function values. Students were expected to use a difference quotient to answer these questions. So students must be provided with the tools to approach problems that are presented as data tables as well as those with analytically defined functions.

### Exploration: Zooming In

The zooming in exploration gives students experience with the idea of derivative as the slope of the curve. After completing this exploration students will be able to estimate the slope of the function (the derivative) at a point by zooming in on that point. Students will also see examples where the slope (the derivative) of a function at a point is not defined. This approach can also be used to reinforce a student's intuitive understanding of limits. This exploration should be scheduled before derivatives are formally defined in class.

Students are expected to have graphing calculators that graph functions in an arbitrary viewing window. Zooming in can be accomplished by adjusting the viewing window or using built-in zooming features of the graphing calculator.

1. Graph the function  $f(x) = x^4 - 10x^2 + 3x$ . Zoom in on the point  $(3, 0)$  until the graph looks like a straight line. Pick a point on the curve other than the point  $(3, 0)$ , but close to  $(3, 0)$ , and estimate the coordinates of the point, keeping at least three decimal places. Calculate the slope of the line through these two points. Use this information to complete the first row of the table in Problem 2.

The number computed above is an approximation to the *slope of the function*  $f(x) = x^4 - 10x^2 + 3x$  at the point  $(3, 0)$ . This slope is also called *the derivative of  $f$  at  $x = 3$* , and is denoted  $f'(3)$ .

2. Use zooming in to estimate the slope of the following functions at the specified points and complete the table below.

Function	Given Point	Chosen Point	Calculated slope
$f(x) = x^4 - 10x^2 + 3x$	(3, 0)		
$f(x) = x^4 - 6x^2$	(1, -5)		
$f(x) = \cos x$	(0, 1)		
$f(x) = \cos x$	$(\frac{\pi}{2}, 0)$		
$f(x) = (x - 1)^{\frac{1}{3}}$	(2, 1)		

This slope of the function  $f$  at the point  $(a, f(a))$  is also called *the derivative of  $f$  at  $a$* , and is denoted  $f'(a)$ .

Compare your calculated slopes with others in your group and give an estimate of the derivatives at the indicated points.

Function	Given Point $a$	Derivate Value $f'(a)$
$f(x) = x^4 - 10x^2 + 3x$	3	
$f(x) = x^4 - 6x^2$	1	
$f(x) = \cos x$	0	
$f(x) = \cos x$	$\frac{\pi}{2}$	
$f(x) = (x - 1)^{\frac{1}{3}}$	2	

- Graph  $f(x) = (x - 1)^{\frac{1}{3}}$  again. This time, zoom in on (1, 0). Describe what you see. By examining your graphs, explain why the slope is not a finite real number at  $x = 1$ . For this function, conclude that  $f'(1)$  does not exist.
- Graph the function  $f(x) = |x^4 - 6x^2|$ . By looking at the graph and zooming in on the points you select, decide at which point the function  $f$  has a derivative and at which points it does not. Support your answers with appropriate sketches.
- What have you learned as a result of completing this lab that you did not know before?

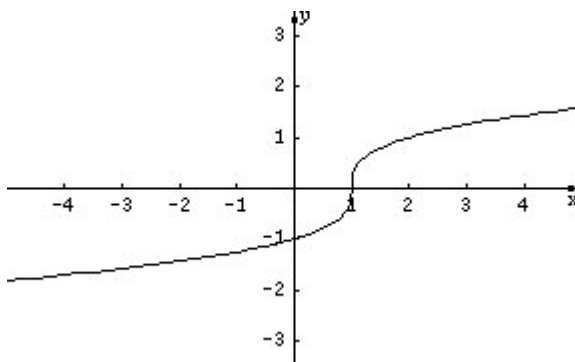
## Teacher Notes

This exploration is adapted from “Lab 3: Zooming In” in *Learning by Discovery: A Lab Manual for Calculus* (Anta Solow, editor), Volume I in the Mathematical Association of America Resources for Calculus Collection (MAA Notes 27).

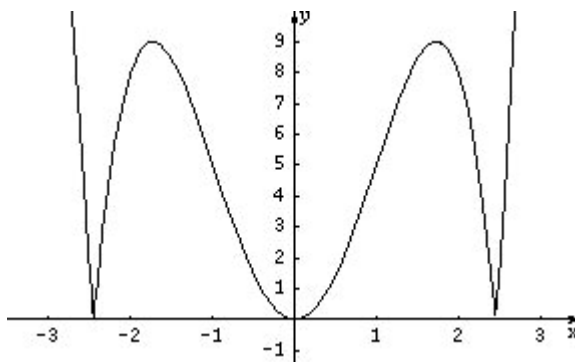
Possible solutions to the tables:

Function	Given Point	Chosen Point	Calculated slope
$f(x) = x^4 - 10x^2 + 3x$	(3, 0)	(3.001..., 0.079...)	51.0687
$f(x) = x^4 - 6x^2$	(1, -5)	(1.0062, -5.0499)	-7.9998
$f(x) = \cos x$	(0, 1)	(0.0015, 0.9999)	0.0007
$f(x) = \cos x$	$(\frac{\pi}{2}, 0)$	(1.575, -0.0042)	-0.9999
$f(x) = (x - 1)^{\frac{1}{3}}$	(2, 1)	(2.00625, 1.0020)	0.3326

Function	Given Point $x = a$	Derivate Value $f'(a)$
$f(x) = x^4 - 10x^2 + 3x$	3	51
$f(x) = x^4 - 6x^2$	1	-8
$f(x) = \cos x$	0	0
$f(x) = \cos x$	$\frac{\pi}{2}$	-1
$f(x) = (x - 1)^{\frac{1}{3}}$	2	1/3



When students zoom in on the point (1, 0), the graph should start to look like a vertical line. Because the slope of a vertical line is undefined, the derivative is not a finite real number.



Students should notice that the graph does not have derivatives at  $x = \pm\sqrt{6}$ . When you zoom in on these  $x$ -values, the graph is not locally linear, i.e., it never looks like a line.

## Estimating Derivatives Numerically

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Gonzaga High School  
Arlington, Virginia

Most students come to AP Calculus with a strong understanding of linear functions. They can find the slope of a line given its equation or given two points on the graph of the line. They likely understand that lines have constant slope. Some may even have used language that includes the phrase *rate of change*. Calculus teachers, therefore, ought to leverage that understanding when they introduce the derivative to students.

In Algebra 1, students are taught that the slope of a line through points  $(x_1, y_1)$  and  $(x_2, y_2)$  is given by

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1},$$

the change in the dependent variable divided by the change in the independent variable.

Using this idea as a starting point, in calculus, students see that the average rate of change of a function  $f$  on a closed interval,  $[a, b]$ , is given by

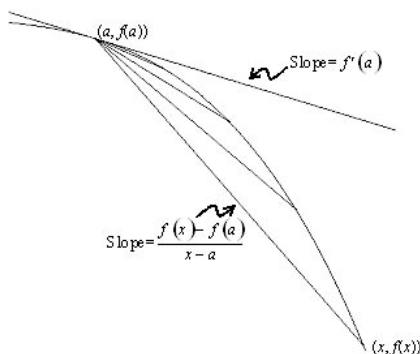
$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a},$$

the change in the function outputs divided by the change in the inputs. Note that the average rate of change of  $f$  on  $[a, b]$  is the slope of the line through the two points  $(a, f(a))$  and  $(b, f(b))$  (sometimes called the secant line).

The landscape changes dramatically in calculus! The idea of limit allows us to determine *instantaneous* rates of change. In fact, the derivative of a function at a point is defined as the *limit* of an average rate of change:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

An illustration like the following, which appears in nearly every calculus textbook, gives a geometric interpretation of the result:



Using a scientific or graphing calculator, students can approximate a derivative by evaluating an average rate of change over a suitably small interval. For example, the derivative of

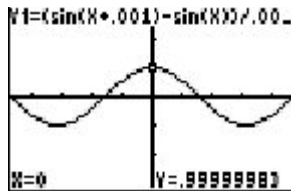
$$f(x) = \sin x \text{ at } x = 0 \text{ is approximately } \frac{\sin(.001) - \sin(0)}{.001 - 0} = 0.999999833.$$

After students have learned about the derivative as a function, and how to differentiate basic functions, try asking them this question:

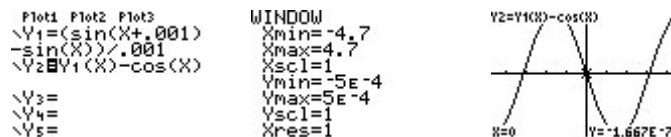
In radian mode, use your calculator to graph  $y1(x) = \frac{\sin(x + .001) - \sin(x)}{.001}$  in a decimal window. Describe the resulting graph, and explain why it looks the way it does.

Since at every value of  $x$ , the expression  $\frac{\sin(x + .001) - \sin(x)}{.001}$  approximates the derivative of  $\sin(x)$ , the graph looks like  $y = \cos(x)$ .

Here's the calculator graph:



So, how good a job does  $y1(x) = \frac{\sin(x + .001) - \sin(x)}{.001}$  do of approximating the derivative of  $\sin(x)$ ? Since at every value of  $x$ , we are using the slope of a secant line to approximate the slope of a tangent line, the *straighter* the graph of  $y = \sin(x)$ , the better the approximation ought to be. How do we measure the straightness of a graph? Along a straight line, slope does not change. From a calculus point of view, the rate of change of slope with respect to  $x$  is 0 for linear functions. In other words, the second derivative of the function is zero. It makes sense, then, that the closer the second derivative is to zero, the better the difference quotient should do in approximating the derivative. It's easy to investigate this on a graphing calculator by plotting the difference between the difference quotient and  $\cos(x)$ :



Amazing! The error has the shape of a negative sine graph, the second derivative of  $\sin(x)$ ! That is, the larger the second derivative, the greater the change of slope of our function, and the less the slope of a secant line approximates the slope of a tangent line. In fact, if we divide this error by the second derivative,  $-\sin(x)$ , we get a function that is close to constant.

**Special Focus: Approximation**

```

Plot1 Plot2 Plot3
\Y1=(sin(X+.001)
-sin(X))/-.001
\Y2=Y1(X)-cos(X)
\Y3=Y2(X)/-sin(X)
)
\Y4=

TABLE SETUP
TblStart=0
ΔTbl=.1
Indent: 0 Ask
Depend: 0 Ask

```

X	Y1
-0.3	SE-4
-0.2	SE-4
-0.1	SE-4
0	ERRDR
.1	SE-4
.2	SE-4
.3	SE-4

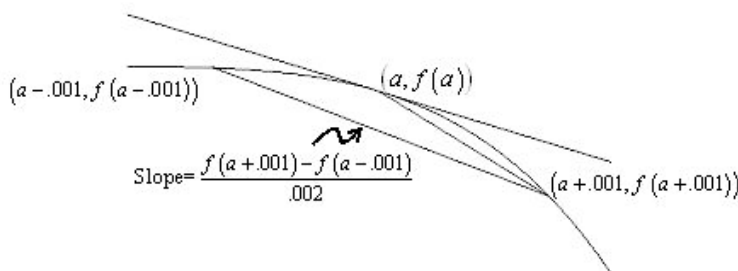
Y3=Y2(X)/-sin(X)

Cool!

How do calculators approximate derivatives? In the absence of a computer algebra system that calculates derivatives symbolically, most machines use a difference quotient to approximate. However, the method used is a *symmetric* difference quotient that spans the point where the derivative is being approximated. The approximation is

$$f'(a) \approx \frac{f(a + .001) - f(a - .001)}{.002}$$

Geometrically, it's using the slope of a secant line that straddles the point where the derivative is being approximated:



The symmetric difference quotient is used to approximate the derivative.

It is just such a symmetric difference quotient that functions like nDeriv or nder on graphing calculators use. Students should understand and be wary of these functions, since they can give unreliable results at or near points where a function is not differentiable. For example, check out the value of nDeriv for the absolute function at  $x = 0$ :

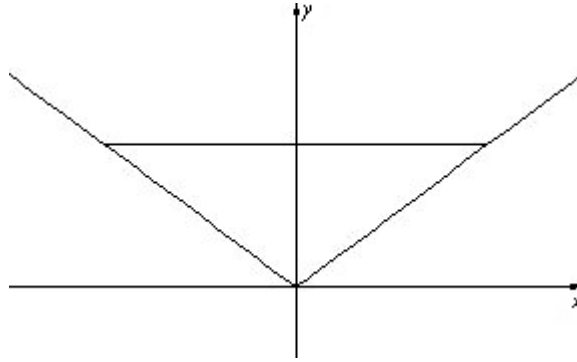
```

nDeriv(abs(X),X,
0)
0

```

Once you know that the machine is using a symmetric difference quotient, the reason behind this error becomes clear.

The symmetric difference quotient is 0 for the absolute value function at  $x = 0$ .



In fact, nDeriv will give a result of 0 for *any* even function at  $x = 0$ ! Here's another example:

```
nDeriv(X^(2/3),X
,0)
      0
```

You might guess that the only points to worry about when using the symmetric difference quotient are places where the function is not differentiable. But the situation is even more hazardous than that. Consider the following examples:

```
nDeriv(abs(X),X,
.0005)
nDeriv(1/X,X,.0005)
      5263157.895
VIKES!
```

The default  $h$  used by the calculator is 0.001. Thus, if you try to evaluate nDeriv anywhere within .001 of a point where a function fails to be differentiable, the result is unreliable. The

second result above comes from evaluating  $\frac{\frac{1}{.0009 + .001} - \frac{1}{.0009 - .001}}{.002}$ .

Of course, a one-sided difference quotient is also prone to error near points where a function's derivative fails to exist. Why did the calculator manufacturers choose the symmetric difference quotient over a one-sided difference quotient? The answer lies in the behavior of the error for each approximation.

Note: This idea of average rate of change, really a precalculus notion, has been tested often on recent AP Exams. (See 2005, AB5 BC5, 2004 AB1 BC1, 2003 Form B AB5 BC5, 1998 AB3, and 1997 AB1 for some examples.)

### Exploration: Errors in the Definitions of Derivative at a Point

Here are function definitions for an activity exploring the behavior in two common approximations to the derivative at a point. In Y1, we define the function whose derivative we are approximating. In Y2, we have the one-sided difference quotient approximation for  $Y1'(A)$ , and in Y3 we have the symmetric difference quotient approximation.

```

Plot1 Plot2 Plot3
\Y1=tan-1(X)
\Y2=(Y1(A+X)-Y1(A))/X
\Y3=(Y1(A+X)-Y1(A-X))/(2X)
\Y4=
\Y5=
    
```

Note that the calculator's independent variable in Function Mode is always  $X$ , and that here it is filling the role of the step size from the definition of derivative (which is usually named  $h$ ). What we really care about is how the errors in these approximations behave. So, in  $Y_4$ , we put the exact value of the derivative,  $Y_1'(A)$ . Then, in  $Y_5$  and  $Y_6$ , we put the errors from using the one-sided and the symmetric difference quotient approximations, respectively. Only  $Y_5$  and  $Y_6$  are selected for graphing.

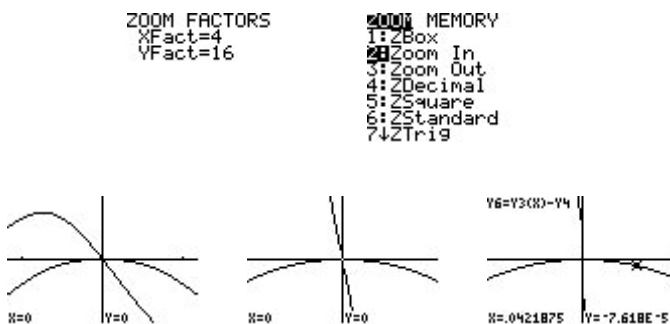
```

Plot1 Plot2 Plot3
\Y4=1/(1+A^2)
\Y5=Y2(X)-Y4
\Y6=Y3(X)-Y4
\Y7=
\Y8=
\Y9=
\Y0=
    
```

Store any real number in the variable  $A$  (0.5 was chosen for the example here).



In the decimal window, there are no apparent differences between the behaviors of the two approximations. But zooming in with a degree-two power zoom (under which functions that exhibit quadratic behavior have graphs that remain unchanged—see the appendix for an explanation of power zooming) is most revealing!



Remember that we are graphing the error in using either the difference quotient (DQ) or the symmetric difference quotient (SDQ) to approximate the derivative of  $f(x) = \arctan(x)$  at  $a = 1/2$  as a function of  $h$ . The results are astounding! The error from using the DQ appears to be linear, and the graph gets steeper as we perform a degree-two power zoom. The error from using the SDQ appears to behave like a quadratic. Its shape remains essentially unchanged as we zoom in with a degree-two power zoom.



You can explore this result numerically by looking at a table of values near  $x = 0$ .

TABLE SETUP			
TblStart=-.003	X	Y5	Y6
ΔTbl=.001	-.003	9.6E-4	-4E-7
IndPnt: Ask	-.002	6.4E-4	-2E-7
Depnd: Ask	-.001	3.2E-4	-4E-8
	0	ERRR	ERRR
	.001	-3E-4	-4E-8
	.002	-6E-4	-2E-7
	.003	-1E-3	-4E-7

$Y5 = \Delta Y_2(X) - Y_4$

Note that Y5, the DQ errors, has a constant difference of about 0.0003, indicating linearity. The rate of change of Y5(X) with respect to X is constant, and this constant is the slope of the line containing the ordered pairs (X,Y5(X)). For Y6 (the SDQ errors) the *second* differences are constant, indicating a quadratic. That is, the rate of change of the rate of change is constant, and this happens with degree-two polynomials (whose second derivatives are constant).

You can verify this result by doing some list operations. Consider the following the calculator screens:

L4	Y5	Y6	# 5
-----	-3E-4	-9E-8	"seq(X,X,.001,.0
	-3E-4	-9E-8	1,.001)"→L1
	-3E-4	-9E-8	seq(X,X,.001,.0...
	-3E-4	-9E-8	"Y5(L1)"→L2
	-3E-4	-9E-8	Y5(L1)
	-3E-4	-9E-8	"Y6(L1)"→L3
	-3E-4	-9E-8	Y6(L1)
L5="ΔList(L2)"			"ΔList(L2)"→L5
			ΔList(L2)
			"ΔList(ΔList(L3))
			"→L6
			ΔList(ΔList(L3))

You can go back and redo this activity with nearly *any* function in Y1, and with *any* value stored in the variable A where Y1 has a derivative. You also need to put the exact derivative value in Y4. It's needed to evaluate the error. The amazing thing is that for nearly any function and nearly any point, the results are the same. Eventually, the error in the SDQ behaves like a quadratic, and the error in the DQ behaves like a line! (It's possible that the DQ errors could behave like a higher-degree polynomial, due to symmetry. For example, the DQ errors for  $f(x) = \sin(x)$  will behave like a quadratic.) Of course, we *prefer* error functions that behave with a higher degree, since higher-degree power functions stay close to zero longer. That's why the calculators use the symmetric difference quotient!

Chances are you would use the symmetric difference if asked to compute a rate of change in real life. For instance, if you knew your odometer readings two hours, three hours, and four hours into a long road trip were, respectively, 100 miles, 170 miles, and 230 miles, and wanted to estimate your velocity (speed) at three hours, you most likely would estimate it as  $(230 - 100)/2$ , the symmetric difference, rather than as  $(230 - 170)/1$ , a one-sided difference. In recent AP Exams, such data has been provided in table form, with students asked to estimate the velocity or other rate of change or derivative from the data. In these problems, students are expected to give the best estimate possible from the data, which generally is the symmetric difference. Examples include 2005 AB3/BC3, part (a), and 2003 AB3 part (a) (see Using Approximations in a Variety of AP Questions).

Note: These activities were inspired by a talk by Don Kreider at TICAP in the early 1990s.

## Appendix: Power Zooming

If you zoom in to the graph of any linear function with equal horizontal and vertical “zoom factors,” the shape of your graph remains unchanged. Try it! First, define any linear function, and set a viewing window where you can see its graph. Draw the graph. Check that your zoom factors are equal. On the TI-83, you do this from the ZOOM Memory menu; on the TI-86, from the ZFACT menu; and on the TI-89, from the ZOOM ... Set Factors ... menu. Press Trace to position the cursor on the pixel halfway across the graph, then zoom in to the graph repeatedly. You should see no change in the slope of the line as you zoom in. Look at the screens below for an example:

```

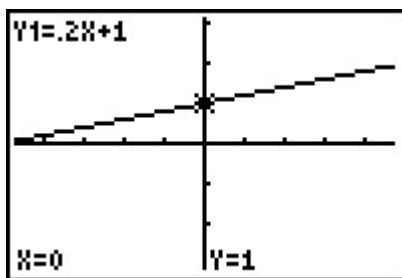
Plot1 Plot2 Plot3
\Y1= .2X+1
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
    
```

```

WINDOW
Xmin=-4.7
Xmax=4.7
Xscl=1
Ymin=-3.1
Ymax=3.1
Yscl=1
Xres=1
    
```

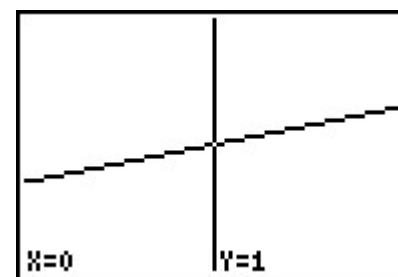
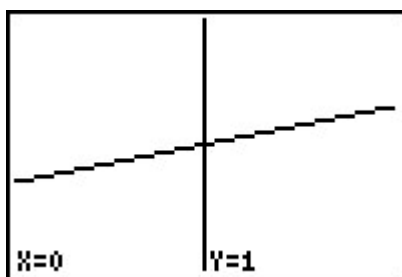
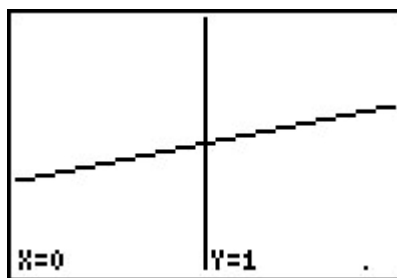
```

ZOOM FACTORS
XFact=4
YFact=4
    
```

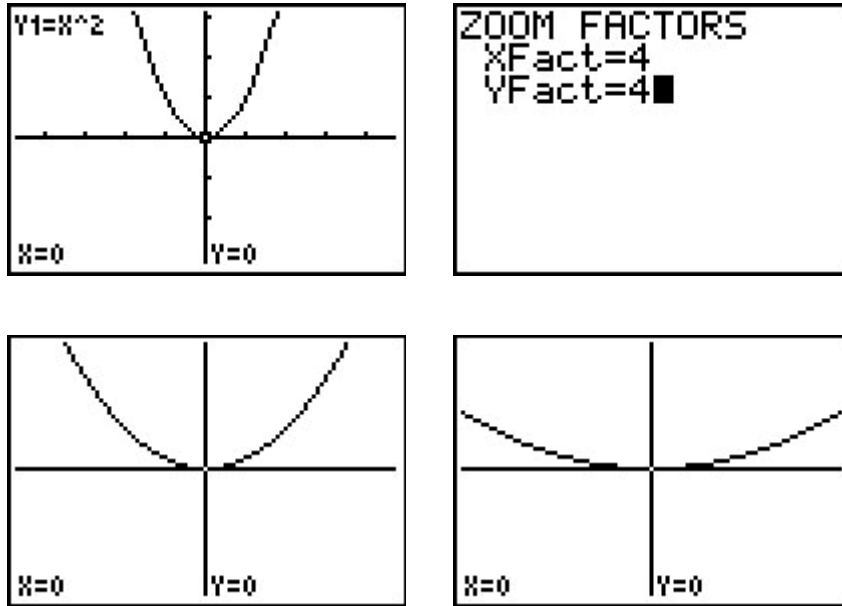


```

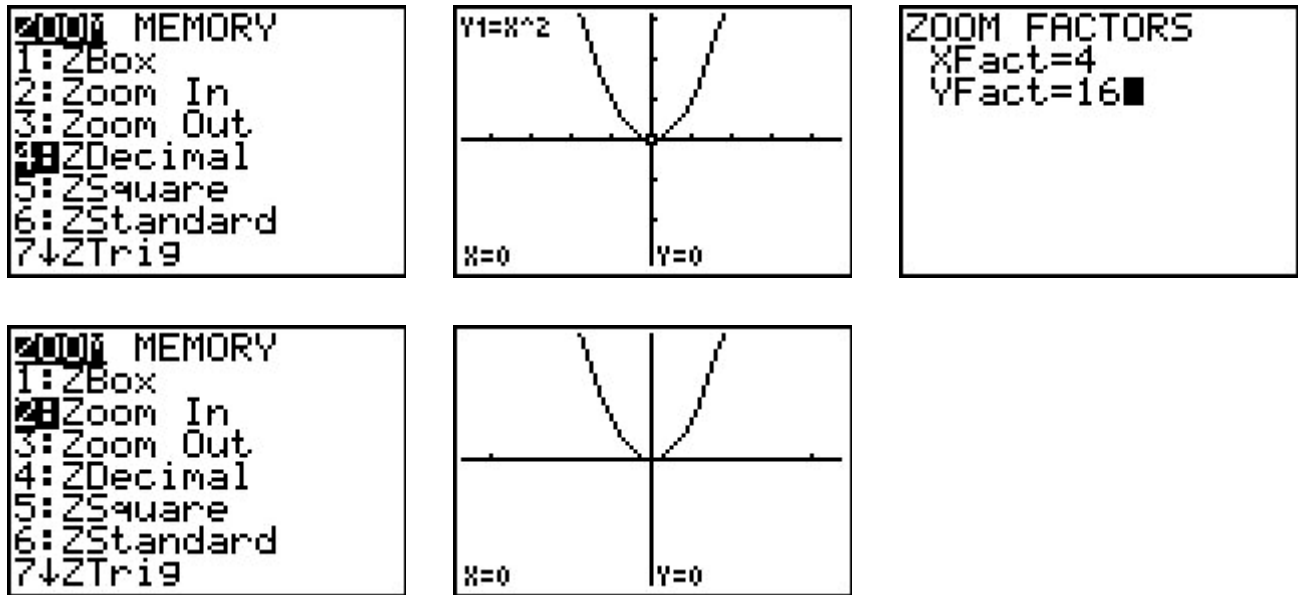
ZOOM MEMORY
1: ZBox
2: Zoom In
3: Zoom Out
4: ZDecimal
5: ZSquare
6: ZStandard
7: ZTrig
    
```



Now, suppose you wanted to zoom in on a parabola (for example,  $y = x^2$ ), and see the same effect: the shape of the graph stays the same as you zoom in. Start at the decimal window (Zoom 4 on the TI-83 and -89, ZOOM ... ZDECM on the -86). Then try zooming in to the graph of  $y = x^2$  with equal zoom factors at (0, 0). You should see the graph flatten out.



Now, change the zoom factors so that the  $y$  factor is the square of the  $x$  factor. For example, try zoom factors of  $X\text{Fact} = 4$  and  $Y\text{Fact} = 16$ . Start again at the decimal window, and try zooming in with these factors. You should see the graph's shape remain unchanged.



This zooming procedure is called a *power zoom of degree two*. The degree is two because the  $y$ -factor is the square of the  $x$ -factor. Now, try zooming in to your graph of  $y = x^2$  with a degree-three power zoom. For example, set  $X\text{Fact} = 2$  and  $Y\text{Fact} = 8$ . Start again from the decimal window, and zoom in a few times with this degree-three power zoom. This time, you should see the graph get steeper.

Let's see if these observations might generalize. Clear out Y1, define  $Y2 = X^2$ ,  $Y3 = X^3$ , and  $Y4 = X^4$ , and set your viewing window to  $[-5, 5]$  horizontally by  $[-25, 25]$  vertically. Leave your zoom factors set for a degree-three power zoom. You might also want to set a different graphing style for Y2, Y3, and Y4 so you can tell the graphs apart. Zoom in a few times to the graph of these three functions.

You should see that the graph of  $y = x^3$  remained unchanged, the graph of  $y = x^2$  got steeper, and the graph of  $y = x^4$  flattened out.

In general, zooming in at the origin to the graph of  $y = x^n$  with a power zoom of degree  $p$  will cause the graph to get steeper if  $n < p$ , stay the same shape if  $n = p$ , and flatten out if  $n > p$ . The same effect will be seen if you zoom in to the graph of  $y = (x - a)^n$  at the point  $(a, 0)$ .

Of course, in all these examples, we already know the degree of each function considered. Power zooming can be used to discover the *degree* of an unknown function. For example, if you zoomed in at a zero to a function with a degree-two power zoom, and the graph got steeper, you'd know it was exhibiting behavior that was less than degree two. Similarly, if the graph flattened out under a degree-two power zoom, then the degree behavior must be greater than two. For some intriguing examples, see the exploration "**Errors in the Definitions of Derivative at a Point**" and the two explorations in the article "**The Tangent Line as 'The Best' Linear Approximation**" for some intriguing examples.

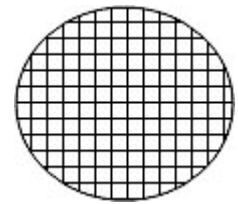
Note: I first saw the idea of power zooming in the calculus textbook by Dick & Patton.

## Approximating Definite Integrals: Riemann Sums as a Tool for Approximation of Areas

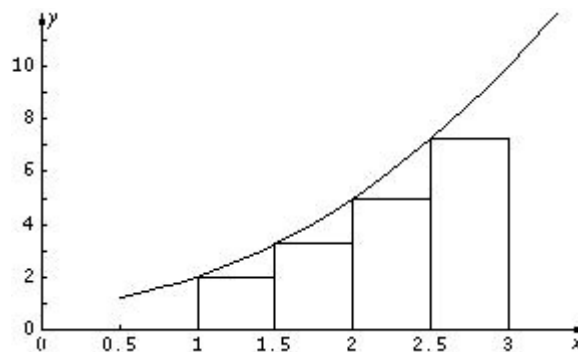
John Jensen  
Rio Salado College  
Tempe, Arizona

Ruth Dover  
Illinois Mathematics and Science Academy  
Aurora, Illinois

It is a common activity for elementary school students to estimate the area of a circle or other nonpolygonal region by superimposing a regular rectilinear grid on the surface and counting the number of entire squares within the boundary of the region. (See the figure to the right.) The approximation of the area bounded by a curve in the plane by using Riemann sums involves a similar idea. By summing the areas of a collection of rectangles (whose dimensions can be readily obtained), we can make a reasonable approximation to the area bounded by a curve (the  $x$ -axis, and the lines  $y = a$  and  $y = b$ ). In addition, we can improve the estimate by increasing the number of rectangles used in the approximation.



Consider the graph of the function  $f(x) = x^2 + 1$  as shown below. Suppose we wish to estimate the area beneath the graph of  $f(x)$  and above the  $x$ -axis from  $x = 1$  to  $x = 3$ . The figure below illustrates one such estimate, called the *left Riemann sum* or *left rectangular sum*. This is formed by using the left-hand endpoint of each subinterval and the corresponding  $y$  value to determine the height of each rectangle.



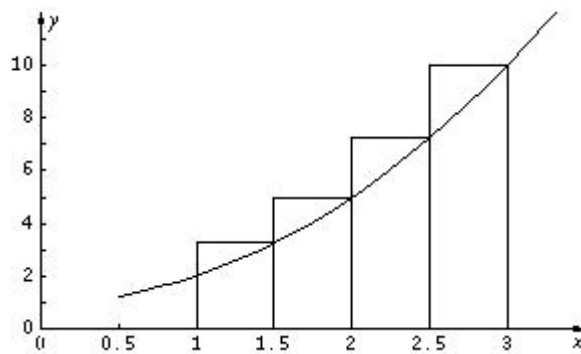
If we regard each subinterval as having length  $\Delta x$ , where  $\Delta x = 1/2$ , then the approximation of the area is:

Special Focus: Approximation

$$\begin{aligned} A &\approx f(1) \cdot \frac{1}{2} + f\left(\frac{3}{2}\right) \cdot \frac{1}{2} + f(2) \cdot \frac{1}{2} + f\left(\frac{5}{2}\right) \cdot \frac{1}{2} \\ &= 2 \cdot \frac{1}{2} + \frac{13}{4} \cdot \frac{1}{2} + 5 \cdot \frac{1}{2} + \frac{29}{4} \cdot \frac{1}{2} \\ &= \frac{35}{4} = 8.75 \end{aligned}$$

It is fairly easy to see from the graph that this approximation is an underestimate. There are pieces of the required area above each of the four rectangles shown. Further, one can see that when the interval is evenly subdivided, the corresponding segments of the curve with the greatest slope result in the greatest amount of area that is unaccounted for. In the example above, there is more area unaccounted for under the curve but above the rectangle from  $x = 2.5$  to  $x = 3$  than there is above the rectangle from  $x = 1$  to  $x = 1.5$ .

Now, consider the *right Riemann sum* or *right rectangular sum* for the same region. This time, each rectangle is formed by using the right-hand endpoint of each subinterval and the corresponding  $y$  value to determine the height of each rectangle.



Once again, if we regard each subinterval as having length  $\Delta x$ , where  $\Delta x = 1/2$ , then the approximation of area is:

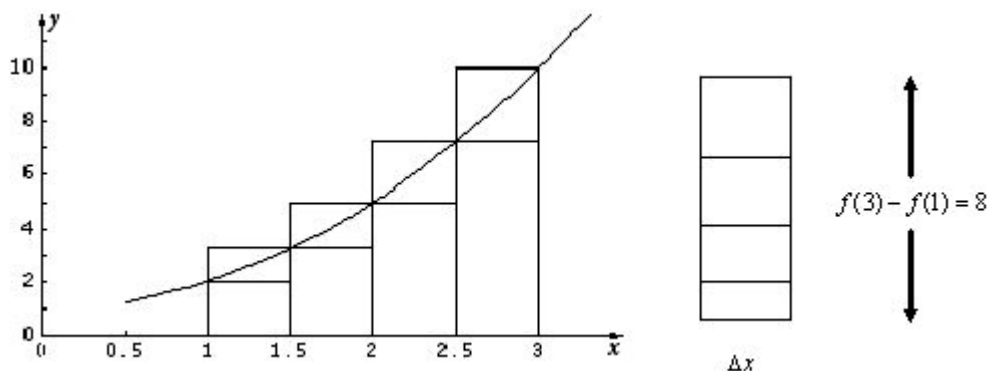
$$\begin{aligned} A &\approx f\left(\frac{3}{2}\right) \cdot \frac{1}{2} + f(2) \cdot \frac{1}{2} + f\left(\frac{5}{2}\right) \cdot \frac{1}{2} + f(3) \cdot \frac{1}{2} \\ &= \frac{13}{4} \cdot \frac{1}{2} + 5 \cdot \frac{1}{2} + \frac{29}{4} \cdot \frac{1}{2} + 10 \cdot \frac{1}{2} \\ &= \frac{51}{4} = 12.75 \end{aligned}$$

It is also clear that this right Riemann sum overestimates the area bounded by the curve and the  $x$ -axis from  $x = 1$  to  $x = 3$ . Once again, the rectangles on the rightmost end of the interval contribute the most to the overestimation. The left and right Riemann sum estimates tell us that  $8.75 < A < 12.75$ .

It is convenient for us to see the underestimate and overestimate together to get some sense of the range of this approximation (see figure below). To the right of the graph, the individual differences of the left and right rectangles are pictured in a *stack*. Because the base of this stack is  $\Delta x$  and the height is  $f(3) - f(1) = 8$ , the area of this assembled

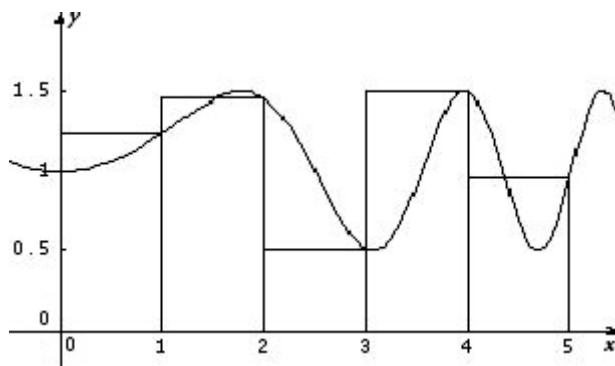
rectangle is  $[f(3) - f(1)] \cdot \Delta x = 4$ . As we increase the number of rectangles, we improve the approximation. Because an increase in the number of rectangles also means that each rectangle is narrower, the entire stack becomes narrower as  $\Delta x \rightarrow 0$ , which is a slightly different way of seeing the result that

$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \cdot \Delta x = \int_a^b f(x) dx$  where the expression  $\int_a^b f(x) dx$  is a *definite integral* that represents the exact area bounded by the curve.

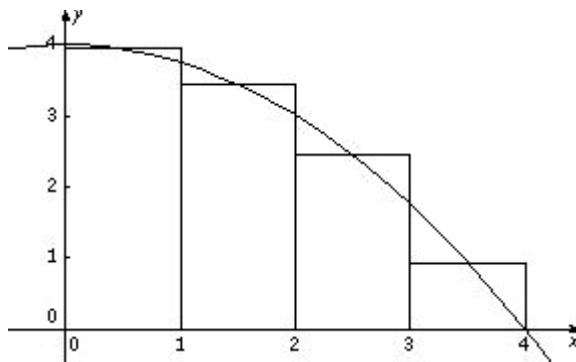


Although we could establish rules to determine whether a left or right Riemann sum for an increasing or decreasing function will overestimate or underestimate the actual area, it is much more instructive to use a sketch of the graph to reach a conclusion. It is also important to note that concavity does not affect this issue. AP Calculus problems frequently ask students to determine whether the approximation is too high or too low, so it is important to consider the graph in order to understand this idea clearly.

Furthermore, there are many cases where it is not clear whether an approximation exceeds or falls short of the actual value. For example, a right Riemann sum applied to the curve  $y = .5\sin(.5x^2) + 1$  (as shown below) leaves us with an unclear picture about whether the sum underestimates or overestimates the actual area (under the curve, above the  $x$ -axis, or between the lines  $x = 0$  and  $x = 5$ ). We might actually discover that the left or right sum comes quite close to the true area because there is a mixture of overestimation and underestimation.



This suggests a third approach for estimating areas using Riemann sums, the *midpoint Riemann sum*. When curves are increasing or decreasing over their domains, this is often the preferred method of estimating the area because it tends to balance *overage* and *underage*. Looking at the curve  $g(x) = 4 - .25x^2$  on the interval  $[0, 4]$ , we see that it is decreasing. Consider the midpoint Riemann sum with four subintervals.



The approximation using a midpoint sum is:

$$\begin{aligned} A &\approx g(.5) \cdot 1 + g(1.5) \cdot 1 + g(2.5) \cdot 1 + g(3.5) \cdot 1 \\ &= 3.9375 + 3.4375 + 2.4375 + .9375 \\ &= 10.75 \end{aligned}$$

The advantage of the midpoint approach in cases like these is evident once we see the graph with its associated rectangles. Plainly, the midpoint rectangles tend to balance the overestimates and underestimates. Students often mistakenly believe that this balance is perfect and that the midpoint approximation is exact. This provides a great opportunity to discuss the role that concavity plays in approximation by Riemann sums.

In the previous example, the midpoint Riemann sum applied to the function  $f(x) = x^2 + 1$  over the interval  $[1, 3]$  provides an estimate of 10.63. While this estimate lies between the left and right approximations, it is not the average (arithmetic mean) of the left and right approximations, which is 10.75. It is instead equivalent to an approximation called the Trapezoidal Sum, which will be discussed later in this article.

## Summary of Riemann Sums

We have three powerful tools for the approximation of areas under a curve in a plane region using rectangles. To summarize symbolically, we need to introduce a little vocabulary and notation.

We begin by partitioning the closed interval  $[a, b]$  in question into  $n$  subintervals with endpoints  $a = x_0, x_1, x_2, \dots, x_n = b$ . If all of the subintervals are formed with equal length,



then  $\Delta x = \frac{b-a}{n}$ , and we often express  $x_i = a + i \cdot \Delta x$ ,  $i = 0, 1, 2, \dots, n$ . This leads to the following symbolic representations.

LEFT RIEMANN SUM:

$$A_L \approx \sum_{i=0}^{n-1} f(x_i) \cdot \Delta x = \sum_{i=1}^n f(x_{i-1}) \cdot \Delta x = f(x_0) \cdot \Delta x + f(x_1) \cdot \Delta x + \dots + f(x_{n-1}) \cdot \Delta x$$

RIGHT RIEMANN SUM:

$$A_R \approx \sum_{i=1}^n f(x_i) \cdot \Delta x = f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \dots + f(x_n) \cdot \Delta x$$

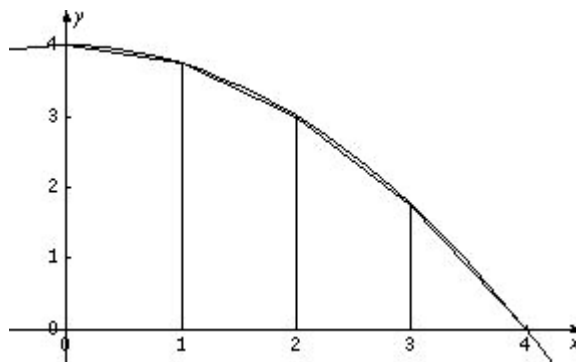
MIDPOINT RIEMANN SUM:

$$A_M \approx \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) \cdot \Delta x = \sum_{i=1}^n f\left(\frac{x_{i-1} + x_i}{2}\right) \cdot \Delta x$$

## The Trapezoidal Rule

The fact that most approximation schemes using Riemann sums leave small, roughly triangular-shaped areas either within or outside the region in question often prompts students to think of a means of including those areas by approximating them with triangles and somehow appending them to the areas obtained. In fact, many students anticipate the Trapezoidal Rule before it is presented as a method for approximating the area of a region in the plane. Aside from being an improvement to the left and right Riemann sums, the Trapezoidal Rule has the added benefit that it relieves students of the decision of whether to use the left or right endpoints to calculate heights, since both of these must be calculated for the trapezoidal rule.

Consider the function  $y = 4 - .25x^2$  on the interval  $[0, 4]$  pictured below:



Note first that the trapezoids are difficult to see simply because the approximation is so good. Because each vertical segment except the left and right ones appears in two trapezoids, the formula involves two of each of them. Notice also that the rightmost trapezoid is actually a

triangle. This has no effect on our calculation. For the diagram shown, the approximation to the area under the curve using trapezoids is:

$$A \approx \frac{f(0) + f(1)}{2} \cdot \Delta x + \frac{f(1) + f(2)}{2} \cdot \Delta x + \frac{f(2) + f(3)}{2} \cdot \Delta x + \frac{f(3) + f(4)}{2} \cdot \Delta x$$

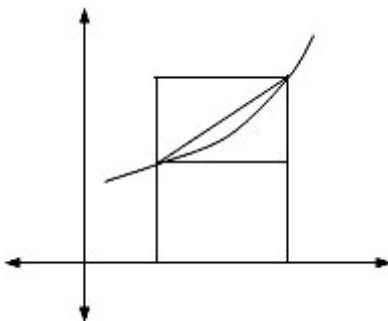
$$= \frac{\Delta x}{2} [f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)]$$

In general, then, the Trapezoidal Rule is simply another tool for estimating area. Often, we refer to  $\Delta x$  as  $h$ , and consider  $n$  trapezoids over the interval  $[a, b]$ , where  $h = \frac{b-a}{n}$ ; thus, the formula appears as:

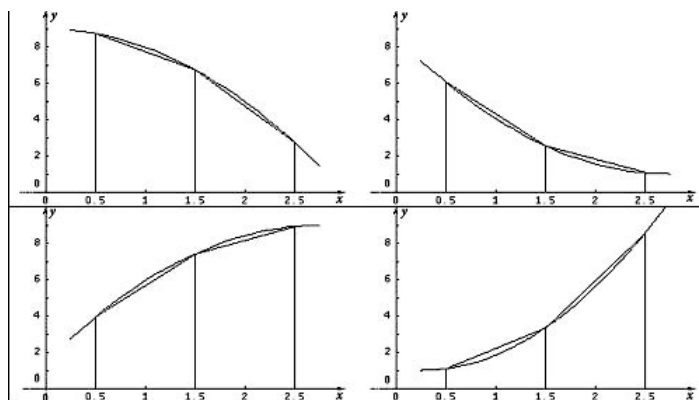
$$A_T \approx \frac{h}{2} [f(a) + 2f(a+h) + 2f(a+2h) + 2f(a+3h) + \dots + 2f(b-h) + f(b)]$$

While many students mistakenly believe that the Midpoint Riemann Sum is the average of the left and right sums, it should be easy to see that the Trapezoidal Rule actually is this average. Using the formulas for the left and right Riemann sums shown above, we get

$A_T = \frac{A_L + A_R}{2}$ . Showing this graphically with just one subinterval should help to convince students of this relationship.



With trapezoidal sums, it is again important to know whether the approximation is too high or too low. A few graphical examples help us to discover the pattern. Consider the graphs below. (Only two trapezoids are shown to show the ideas more clearly.)



Two of the figures show graphs of decreasing functions, and two show graphs of increasing functions. Two are concave down while the other two are concave up.

Under what conditions is the approximation too high? Too low? Many people initially guess that whether the function increases or decreases affects the size of the approximation relative to the actual value, but this is not the case. It is only the concavity of the graph that determines whether the approximation is too high or too low. If the graph of the function is concave up, then the approximation is too high since the straight lines of the trapezoid are above the graph. Similarly, if the graph of the function is concave down, then the trapezoidal approximation is too low. Once again, this pattern should not be memorized by students. It is much easier to sketch a graph of the function and one trapezoid to observe the pattern and answer the question.

### Approximations with Unequal Subintervals

When dealing with real-world data, the subdivisions, or subintervals, are often of unequal length. On recent AP Calculus Exams, problems such as these have been particularly difficult for students. This may be because students view Riemann and trapezoidal sums as rules rather than concepts, which is what these sums actually represent. Consider the following example in which the values of a function are given at unequal subintervals.

$x$	0	2	3	7	9
$f(x)$	3	6	7	6	8

Note how the left- and right-hand approximations below are formed.

$$LH = 3 \cdot 2 + 6 \cdot 1 + 7 \cdot 4 + 6 \cdot 2 = 52$$

$$RH = 6 \cdot 2 + 7 \cdot 1 + 6 \cdot 4 + 8 \cdot 2 = 59$$

And the trapezoidal approximation.

$$Trap = \frac{1}{2}(3 + 6) \cdot 2 + \frac{1}{2}(6 + 7) \cdot 1 + \frac{1}{2}(7 + 6) \cdot 4 + \frac{1}{2}(6 + 8) \cdot 2 = \frac{111}{2} = 55.5$$

Or, equivalently,  $Trap = (LH + RH)/2 = 55.5$

## Instructional Unit: Left, Right, and Midpoint Riemann Sums

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### Context

#### Entry Behaviors

Students should be able to use their calculators to perform arithmetical computations, and be able to evaluate standard functions at particular points. Students should also be able to construct graphs of these standard functions.

#### Prior Knowledge

Students should understand the definition of the definite integral as a limit of Riemann sums. They should also understand that the definite integral gives the area between the graph of a function and the  $x$ -axis when that function lies entirely above the  $x$ -axis on the interval of integration. Students should have encountered the Fundamental Theorem of Calculus and be able to compute definite integrals using that theorem. In addition, students should know the Mean Value Theorem and understand the idea of linear approximation to a function at a point.

#### Academic Motivation

Students should have encountered integrals for which the Fundamental Theorem of Calculus does not provide a viable way of computing a definite integral. The motivation for this unit is to be able to approximate definite integrals because the Fundamental Theorem does not provide a way to evaluate them exactly.

#### Education and Ability

Students should have taken the typical high school mathematics courses of Algebra I and Algebra II, and have a course that covers precalculus material such as trigonometry and conceptual ideas surrounding functions.

### Performance Setting

#### Social Aspects

This should be used in an AP Calculus classroom where a teacher is guiding the exploration of ideas.

**Physical Aspects**

The teacher should have the ability to display visual and written information. Each student should have his or her own graphing calculator.

**Relevance**

This unit is relevant to anyone needing to evaluate definite integrals whose integrands are given in symbolic form or are given by a data table.

**Duration**

This unit should take three to five days to complete.

**Accessibility and Adaptability**

This unit is accessible to anyone with the prior knowledge and a computational device. While graphing calculators are the primary form of this device, anyone with computational software could adapt this lesson to his or her setting. Students with a scientific calculator could do most of the work here but might not be able to explore the more geometric parts of the lesson.

**Goals and Standards****Essential Question**

The essential question addressed here is how to evaluate/approximate an integral when the Fundamental Theorem of Calculus does not provide a viable way of doing so.

**Goals**

The goal is for students to be able to use left, right, and midpoint Riemann sums to approximate the value of a definite integral. They should also be able to estimate the maximum possible error that they have made with their approximation.

**Alignment with the AP Calculus Syllabus**

Students are required by the AP Calculus syllabus to be able to use left, right, and midpoint Riemann sums to approximate a definite integral. However, they are not required to be able to do an error analysis.

**Left, Right, and Midpoint Sum Approximations****Introduction**

1. Motivation — The integral  $\int_{-1}^1 e^{-x^2} dx$  cannot be evaluated exactly. Use this example to develop the following ideas.

2. Recall the definition of the definite integral:

$$\int_a^b f(x)dx = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k)\Delta x_k$$

where  $P : a = x_0 < x_1 < \dots < x_n = b$  is a partition of the interval  $[a, b]$ ,  $c_k$  is an arbitrary point in the  $k^{th}$  subinterval  $[x_{k-1}, x_k]$ ,  $\Delta x_k = x_k - x_{k-1}$  is the length of the  $k^{th}$  subinterval, and  $\|P\| = \max(\Delta x_1, \Delta x_2, \dots, \Delta x_n)$  is the length of the longest subinterval. The summation on the right side of the equation is called a Riemann sum.

3. When  $\|P\| \approx 0$  then  $\int_a^b f(x)dx \approx \sum_{k=1}^n f(c_k)\Delta x_k$ . Thus, Riemann sums can be used to approximate a definite integral.

### Left Riemann Sums

4. Left sums — Choose  $c_k$  to be the left endpoint of the  $k^{th}$  subinterval for each  $k$  ( $c_k = x_{k-1}$ ). Thus we will have  $\int_a^b f(x)dx \approx \sum_{k=1}^n f(x_{k-1})\Delta x_k$ . The summation is called the left Riemann sum approximation of the definite integral.
5. Either give students a partition of the interval  $[-1, 1]$  or have them choose their own partition. Then have them construct a table of the following form to approximate

$\int_{-1}^1 e^{-x^2} dx$  using a left Riemann sum. See Appendix 1 for a table template.

$k$	$k^{th}$ subinterval $[x_{k-1}, x_k]$	left endpoint $x_{k-1}$	$f(x_{k-1}) = e^{-(x_{k-1})^2}$	$\Delta x_k$	$f(x_{k-1})\Delta x_k$
1	$[-1, x_1]$	-1	$e^{-1} \approx 0.367879$	$x_1 - (-1)$	$0.367879(x_1 + 1)$
2	$[x_1, x_2]$	$x_1$	$f(x_1) = e^{-(x_1)^2}$	$x_2 - x_1$	$e^{-(x_1)^2} (x_2 - x_1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n$	$[x_{n-1}, 1]$	$x_{n-1}$	$f(x_{n-1}) = e^{-(x_{n-1})^2}$	$1 - x_{n-1}$	$e^{-(x_{n-1})^2}(1 - x_{n-1})$
				<b>Sum</b>	$\sum_{k=1}^n f(x_{k-1})\Delta x_k$

Students should be able to compute these Riemann sums by hand for a small number of subintervals with a calculator to do the necessary arithmetic. To find more useful approximations using smaller (and more) subintervals, students could program their calculators or use a spreadsheet program such as Excel to automate the necessary arithmetic. A particular example of such a table is:

$k$	$k^{\text{th}}$ subinterval $[x_{k-1}, x_k]$	left endpoint $x_{k-1}$	$f(x_{k-1}) = e^{-(x_{k-1})^2}$	$\Delta x_k$	$f(x_{k-1})\Delta x_k$
1	$[-1.00, -0.80]$	$-1.00$	0.367879	0.20	0.073575888
2	$[-0.80, -0.75]$	$-0.80$	0.527292	0.05	0.026364621
3	$[-0.75, -0.50]$	$-0.75$	0.569783	0.25	0.142445706
4	$[-0.50, -0.30]$	$-0.50$	0.778801	0.20	0.155760157
5	$[-0.30, 0.00]$	$-0.30$	0.913931	0.30	0.274179356
6	$[0.00, 0.10]$	$0.00$	1.000000	0.10	0.100000000
7	$[0.10, 0.30]$	$0.10$	0.990050	0.20	0.198009967
8	$[0.30, 0.65]$	$0.30$	0.913931	0.35	0.319875915
9	$[0.65, 0.85]$	$0.65$	0.655406	0.20	0.131081252
10	$[0.85, 1.00]$	$0.85$	0.485537	0.15	0.072830534
<b>Sum</b>					1.494123395

If we use the calculator to find a decimal approximation to the integral, we get

$$\int_{-1}^1 e^{-x^2} dx \approx 1.493648266.$$

### Right Riemann Sums

6. The technique is similar to left Riemann sums except that  $c_k$  is chosen to be the right endpoint ( $c_k = x_k$ ) of each subinterval, rather than the left endpoint. Again give students a partition of the interval  $[-1, 1]$ , have them use the one they used for a left sum, or have them use a new partition of their own choosing. Have them construct a table like the one they did for the left Riemann sum.

$k$	$k^{\text{th}}$ subinterval $[x_{k-1}, x_k]$	right endpoint $x_k$	$f(x_k) = e^{-(x_k)^2}$	$\Delta x_k$	$f(x_k)\Delta x_k$
1	$[-1, x_1]$	$x_1$	$f(x_1) = e^{-(x_1)^2}$	$x_1 - (-1)$	$e^{-(x_1)^2} (x_1 - (-1))$
2	$[x_1, x_2]$	$x_2$	$f(x_2) = e^{-(x_2)^2}$	$x_2 - x_1$	$e^{-(x_2)^2} (x_2 - x_1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n$	$[x_{n-1}, 1]$	$x_n$	$e^{-1} \approx 0.367879$	$1 - x_{n-1}$	$0.367879(1 - x_{n-1})$
<b>Sum</b>					$\sum_{k=1}^n f(x_k)\Delta x_k$

Using the same partition used in part 5, we have the following table.

$k$	$k^{\text{th}}$ subinterval $[x_{k-1}, x_k]$	right endpoint $x_k$	$f(x_k) = e^{-(x_k)^2}$	$\Delta x_k$	$f(x_k)\Delta x_k$
1	$[-1.00, -0.80]$	-0.80	0.527292	0.20	0.105458485
2	$[-0.80, -0.75]$	-0.75	0.569783	0.05	0.028489141
3	$[-0.75, -0.50]$	-0.50	0.778801	0.25	0.194700196
4	$[-0.50, -0.30]$	-0.30	0.913931	0.20	0.182786237
5	$[-0.30, 0.00]$	0.00	1.000000	0.30	0.300000000
6	$[0.00, 0.10]$	0.10	0.990050	0.10	0.099004983
7	$[0.10, 0.30]$	0.30	0.913931	0.20	0.182786237
8	$[0.30, 0.65]$	0.65	0.655406	0.35	0.229392189
9	$[0.65, 0.85]$	0.85	0.485537	0.20	0.097107379
10	$[0.85, 1.00]$	1.00	0.367879	0.15	0.055181916
				<b>Sum</b>	1.474906764

You can also find Riemann sum applets at the following sites to illustrate ideas geometrically.

- a) <http://www.slu.edu/classes/maymk/Riemann/Riemann.html>
- b) <http://science.kennesaw.edu/~plaval/applets/Riemann.html>
- c) <http://www.calvin.edu/~rpruim/courses/m161/F01/java/RiemannSums.shtml>
- d) <http://www.math.tamu.edu/AppliedCalc/Classes/Riemann/>
- e) <http://www.csun.edu/~hcmth018/RS.html>

Only this last applet allows one to use subintervals that don't have equal width. However, the user has no control over the subintervals. They are chosen randomly by the applet.

### Midpoint Riemann Sums

7. In order to compute a midpoint Riemann sum,  $c_k$  is chosen to be the midpoint

$\left(c_k = \frac{x_{k-1} + x_k}{2}\right)$  of each subinterval, rather than one of the endpoints. Again give students a partition of the interval  $[-1, 1]$ , have them use the one they used for a left sum, or have them use a new partition of their own choosing. Have them construct a table like the one they did for the left and right Riemann sum.



$k$	$k^{\text{th}}$ subinterval $[x_{k-1}, x_k]$	midpoint $c_k = \frac{x_{k-1} + x_k}{2}$	$f(c_k) = e^{-(c_k)^2}$	$\Delta x_k$	$f(c_k)\Delta x_k$
1	$[-1, x_1]$	$c_1 = \frac{-1 + x_1}{2}$	$f(c_1) = e^{-(c_1)^2}$	$x_1 - (-1)$	$e^{-(c_1)^2} (x_1 - (-1))$
2	$[x_1, x_2]$	$c_2 = \frac{x_1 + x_2}{2}$	$f(c_2) = e^{-(c_2)^2}$	$x_2 - x_1$	$e^{-(c_2)^2} (x_2 - x_1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n$	$[x_{n-1}, 1]$	$c_n = \frac{x_{n-1} + x_n}{2}$	$f(c_n) = e^{-(c_n)^2}$	$1 - x_{n-1}$	$e^{-(c_n)^2} (1 - x_{n-1})$
<b>Sum</b>					$\sum_{k=1}^n f(c_k)\Delta x_k$

Using a regular partition with each  $\Delta x_k = 0.2$  ( $n = 10$  subintervals), we have the following table for a midpoint Riemann sum.

$k$	$k^{\text{th}}$ subinterval $[x_{k-1}, x_k]$	midpoint $c_k = \frac{x_{k-1} + x_k}{2}$	$f(c_k) = e^{-(c_k)^2}$	$\Delta x_k$	$f(c_k)\Delta x_k$
1	$[-1.0, -0.8]$	-0.9	0.444858	0.2	0.088971613
2	$[-0.8, -0.6]$	-0.7	0.612626	0.2	0.122525279
3	$[-0.6, -0.4]$	-0.5	0.778801	0.2	0.155760157
4	$[-0.4, -0.2]$	-0.3	0.913931	0.2	0.182786237
5	$[-0.2, 0.0]$	-0.1	0.990050	0.2	0.198009967
6	$[0.0, 0.2]$	0.1	0.990050	0.2	0.198009967
7	$[0.2, 0.4]$	0.3	0.913931	0.2	0.182786237
8	$[0.4, 0.6]$	0.5	0.778801	0.2	0.155760157
9	$[0.6, 0.8]$	0.7	0.612626	0.2	0.122525279
10	$[0.8, 1.0]$	0.9	0.444858	0.2	0.088971613
<b>Sum</b>					1.496106505

### Exploration of Properties

8. The applets can be used to explore the following properties of these sums.
  - a) A left Riemann sum for a function increasing on the interval will be an underapproximation.
  - b) A left Riemann sum for a function decreasing on the interval will be an overapproximation.

- c) A right Riemann sum for a function increasing on the interval will be an overapproximation.
  - d) A right Riemann sum for a function decreasing on the interval will be an underapproximation.
9. Try to challenge students to discover these properties by using functions that are either strictly increasing or decreasing on the intervals. Ask them to construct tables as before, along with a picture similar to those found in the applets.

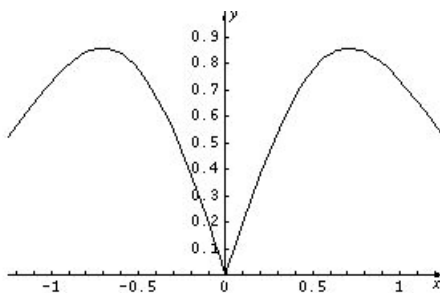
**Error Estimates**

While error estimates are not a part of the AP syllabus, numerical approximations of integrals don't have much worth unless we know what kind of errors are being made in the approximation.

10. Error formulas for the three methods above are given by the formulas below. These formulas are valid when using subintervals of equal width. Proofs for these formulas can be found in Appendix 2. In all of the formulas below  $M_1 = \max_{a \leq x \leq b} (|f'(x)|)$ ,  $M_2 = \max_{a \leq x \leq b} (|f''(x)|)$ , and E represents the error in the approximation by that sum. The value of  $M_1$  can most easily be found by plotting the function defined by  $y = |f'(x)|$  on the interval of interest.

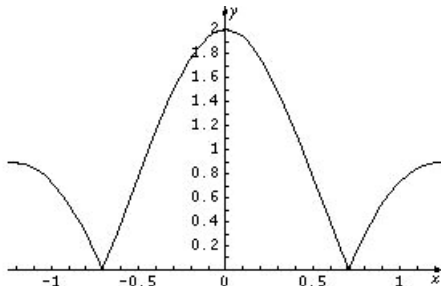
Riemann Sum	Error
Left	$ E  \leq \frac{M_1(b-a)^2}{2n}$
Right	$ E  \leq \frac{M_1(b-a)^2}{2n}$
Midpoint	$ E  \leq \frac{M_2(b-a)^3}{24n^2}$

11. For the example above,  $\int_{-1}^1 e^{-x^2} dx$ , suppose we had used 10 subintervals of equal width  $\Delta x = \frac{1 - (-1)}{10} = \frac{1}{5} = .2$  in all of our sums. For  $f(x) = e^{-x^2}$  we have  $f'(x) = -2xe^{-x^2}$ . Graphing  $y = |f'(x)|$  on the interval  $[-1, 1]$  we get



From this graph we can simply observe that  $M_1 \leq 0.90$ . One could use more sophistication and get a better approximation to  $M_1$ , but it's probably better to get a

quick upper bound for  $M_1$  rather than a more careful approximation. Thus a left or right Riemann sum approximation has an error  $E$  with  $|E| \leq \frac{0.90 \cdot (2)^2}{2.10} = 0.180$ . For the midpoint approximation we got in part 7) we first find  $f''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$ . Graphing  $y = |f''(x)|$  we get



and can see that  $M_2 = 2$ . Using this we get a midpoint approximation error  $E_{mid}$  with  $|E_{mid}| \leq \frac{2 \cdot (2)^3}{24 \cdot 10^2} \leq 0.006666667$ . Since our midpoint approximation was 1.496106505 from Part 7) we now know that

$$1.496106505 - 0.006666667 < \int_{-1}^1 e^{-x^2} dx < 1.496106505 + 0.006666667$$

or equivalently,  $1.489439838 < \int_{-1}^1 e^{-x^2} dx < 1.502773172$ .

Students should be given the chance to do this kind of error analysis so that they have both an upper bound and lower bound on the integral they're trying to evaluate. One should note that the idea of an upper and lower bound have already been introduced already in terms of over- and underapproximations for functions that are strictly increasing or decreasing on the interval of integration. For example, any right

Riemann sum for  $\int_a^b f(x)dx$  gives an upper bound for this integral when  $f$  is increasing on the interval  $[a, b]$ .

12. We can also use these errors to realize a desired accuracy for an integral. Suppose, for example, we had wished to achieve an error for the midpoint approximation so that

$$|E_{mid}| < 0.001. \text{ We can achieve this by making } \frac{M_2(b-a)^3}{24n^2} \leq 0.001. \text{ This is equivalent to}$$

$$\sqrt{\frac{M_2(b-a)^3}{24 \cdot (0.001)}} \leq n. \text{ For this example this would mean that}$$

$$n \geq \sqrt{\frac{M_2(b-a)^3}{24 \cdot (0.001)}} = \sqrt{\frac{2(2)^3}{24 \cdot (0.001)}} \approx 25.82. \text{ Thus we would need a regular partition using at least 26 subintervals to achieve the desired error. To guarantee this maximum error of 0.001 using a left or right sum for this integral, we would need } \frac{M_1(b-a)^2}{2n} \leq 0.001 \text{ or}$$

$$n \geq \frac{.96 \cdot (2)^2}{2 \cdot (0.001)} = 1,920.$$

13. Appendix 3 contains worksheets for students to complete using these ideas. Solutions to the problems in these worksheets are found in Appendix 4.

## Appendix 1: Riemann Sum Table Templates

$k$	$k^{\text{th}}$ subinterval $[x_{k-1}, x_k]$	left endpoint $x_{k-1}$	$f(x_{k-1})$	$\Delta x_k$	$f(x_{k-1})\Delta x_k$
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					
11					
12					
13					
14					
15					
16					
17					
18					
19					
20					
				<b>Sum</b>	

Instructional Unit: Left, Right, and Midpoint Riemann Sums

$k$	$k^{\text{th}}$ subinterval $[x_{k-1}, x_k]$	right endpoint $x_k$	$f(x_k)$	$\Delta x_k$	$f(x_k)\Delta x_k$
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					
11					
12					
13					
14					
15					
16					
17					
18					
19					
20					
				<b>Sum</b>	

**Special Focus: Approximation**

$k$	$k^{\text{th}}$ subinterval $[x_{k-1}, x_k]$	midpoint $\frac{x_{k-1} + x_k}{2}$	$f\left(\frac{x_{k-1} + x_k}{2}\right)$	$\Delta x_k$	$f\left(\frac{x_{k-1} + x_k}{2}\right)\Delta x_k$
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					
11					
12					
13					
14					
15					
16					
17					
18					
19					
20					
				<b>Sum</b>	

## Appendix 2: Error Bounds for Riemann Sum Approximations

Proposition:

Let  $\sum_{k=1}^n f(x_{k-1})\Delta x$  where  $\Delta x = \frac{b-a}{n}$  be the left Riemann sum approximation of  $\int_a^b f(x)dx$ .

If  $E = \int_a^b f(x)dx - \sum_{k=1}^n f(x_{k-1})\Delta x$  then  $|E| \leq \frac{M_1(b-a)^2}{2n}$  if  $f$  is continuously differentiable on  $[a, b]$ .

**Proof:**

Consider the  $k^{\text{th}}$  subinterval  $[x_{k-1}, x_k]$ . Using the Mean Value Theorem, we have that for each  $x$  in this interval there is a point  $c_x$  with  $x_{k-1} \leq c_x \leq x$  so that  $f(x) = f(x_{k-1}) + f'(c_x)(x - x_{k-1})$ . Therefore

$$\begin{aligned} \int_{x_{k-1}}^{x_k} f(x)dx &= \int_{x_{k-1}}^{x_k} (f(x_{k-1}) + f'(c_x)(x - x_{k-1}))dx \\ &= \int_{x_{k-1}}^{x_k} f(x_{k-1})dx + \int_{x_{k-1}}^{x_k} f'(c_x)(x - x_{k-1})dx \\ &= f(x_{k-1})(x_k - x_{k-1}) + \int_{x_{k-1}}^{x_k} f'(c_x)(x - x_{k-1})dx \\ &= f(x_{k-1})\Delta x + \int_{x_{k-1}}^{x_k} f'(c_x)(x - x_{k-1})dx \end{aligned}$$

Thus  $\int_{x_{k-1}}^{x_k} f(x)dx - f(x_{k-1})\Delta x = \int_{x_{k-1}}^{x_k} f'(c_x)(x - x_{k-1})dx$  and

$$\begin{aligned} \int_a^b f(x)dx - \sum_{k=1}^n f(x_{k-1})\Delta x &= \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(x)dx - f(x_{k-1})\Delta x \\ &= \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f'(c_x)(x - x_{k-1})dx \end{aligned}$$

From this

$$\begin{aligned} |E| &= \left| \int_a^b f(x)dx - \sum_{k=1}^n f(x_{k-1})\Delta x \right| \\ &= \left| \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f'(c_x)(x - x_{k-1})dx \right| \end{aligned}$$

$$\begin{aligned} &\leq \sum_{k=1}^n \left| \int_{x_{k-1}}^{x_k} f'(c_x)(x - x_{k-1}) dx \right| \\ &\leq \sum_{k=1}^n \int_{x_{k-1}}^{x_k} |f'(c_x)(x - x_{k-1})| dx \\ &\leq \sum_{k=1}^n \int_{x_{k-1}}^{x_k} M_1(x - x_{k-1}) dx \end{aligned}$$

or now

$$\begin{aligned} |E| &\leq \sum_{k=1}^n M_1 \left( \frac{1}{2}(x - x_{k-1})^2 \right) \Big|_{x_{k-1}}^{x_k} \\ &= \sum_{k=1}^n M_1 \left( \frac{1}{2}(x_k - x_{k-1})^2 \right) \\ &= nM_1 \frac{(\Delta x)^2}{2} \\ &= nM_1 \frac{\left(\frac{b-a}{n}\right)^2}{2} \\ &= \frac{M_1(b-a)^2}{2n} \end{aligned}$$

**Proposition:**

Let  $\sum_{k=1}^n f(x_k)\Delta x$  where  $\Delta x = \frac{b-a}{n}$  be the right Riemann sum approximation of  $\int_a^b f(x)dx$ .

If  $E = \int_a^b f(x)dx - \sum_{k=1}^n f(x_k)\Delta x$  then  $|E| \leq \frac{M_1(b-a)^2}{2n}$  where  $M_1 = \max_{a \leq x \leq b} (|f'(x)|)$  if  $f$  is continuously differentiable on  $[a, b]$ , i.e. if the derivative of  $f$  is continuous on  $[a, b]$ .

**Proof:**

The proof here is essentially identical to the proof above. The only difference is this one uses the Mean Value Theorem to get a point  $c_x$  with  $x \leq c_x \leq x_k$  so that  $f(x) = f(x_k) + f'(c_x)(x - x_k)$ .

**Proposition:**

Let  $\sum_{k=1}^n f\left(\frac{x_{k-1} + x_k}{2}\right)\Delta x$  where  $\Delta x = \frac{b-a}{n}$  be the midpoint Riemann sum approximation of  $\int_a^b f(x)dx$ .

If  $E = \int_a^b f(x)dx - \sum_{k=1}^n f\left(\frac{x_{k-1} + x_k}{2}\right)\Delta x$  then  $|E| \leq \frac{M_2(b-a)^3}{24n^2}$  where  $M_2 = \max_{a \leq x \leq b} (|f''(x)|)$  if  $f$  has a second derivative that is continuous on  $[a, b]$ .



**Proof:**

Consider the  $K^{\text{th}}$  subinterval  $[x_{k-1}, x_k]$  again. One can show using the Mean Value Theorem that we have for each  $x$  in this interval a point  $c_x$  between  $x$  and  $m_k = \frac{x_{k-1} + x_k}{2}$  so that

$$f(x) = f(m_k) + f'(m_k)(x - m_k) + \frac{f''(c_x)}{2}(x - m_k)^2.$$

Therefore, similar to the first proposition,

$$\begin{aligned} \int_{x_{k-1}}^{x_k} f(x) dx &= \int_{x_{k-1}}^{x_k} \left( f(m_k) + f'(m_k)(x - m_k) + \frac{f''(c_x)}{2}(x - m_k)^2 \right) dx \\ &= \int_{x_{k-1}}^{x_k} f(m_k) dx + \int_{x_{k-1}}^{x_k} f'(m_k)(x - m_k) dx + \int_{x_{k-1}}^{x_k} \frac{f''(c_x)}{2}(x - m_k)^2 dx \\ &= f(m_k)(x_k - x_{k-1}) + f'(m_k) \frac{(x - m_k)^2}{2} \Big|_{x_{k-1}}^{x_k} + \int_{x_{k-1}}^{x_k} \frac{f''(c_x)}{2}(x - m_k)^2 dx \\ &= f(m_k)\Delta x + 0 + \int_{x_{k-1}}^{x_k} \frac{f''(c_x)}{2}(x - m_k)^2 dx \end{aligned}$$

The second integral in the second line above is 0 because  $(x_k - m_k)^2 = (x_{k-1} - m_k)^2$  as  $m_k$  is the midpoint of each subinterval. Using this, we now get

$$\begin{aligned} |E| &= \left| \int_a^b f(x) dx - \sum_{k=1}^n f(m_k)\Delta x \right| \\ &= \left| \sum_{k=1}^n \left( \int_{x_{k-1}}^{x_k} f(x) dx - f(m_k)\Delta x \right) \right| \\ &= \left| \sum_{k=1}^n \left( \int_{x_{k-1}}^{x_k} \frac{f''(c_x)}{2}(x - m_k)^2 dx \right) \right| \\ &\leq \sum_{k=1}^n \left| \int_{x_{k-1}}^{x_k} \frac{f''(c_x)}{2}(x - m_k)^2 dx \right| \\ &\leq \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \frac{|f''(c_x)|}{2}(x - m_k)^2 dx \\ &\leq \sum_{k=1}^n \frac{M_2}{2} \int_{x_{k-1}}^{x_k} (x - m_k)^2 dx \\ &= \sum_{k=1}^n \frac{M_2}{2} \frac{(x - m_k)^3}{3} \Big|_{x_{k-1}}^{x_k} \end{aligned}$$

**Special Focus: Approximation**

$$= \sum_{k=1}^n \frac{M_2}{2} \frac{2(x_k - m_k)^3}{3}$$

$$= n \frac{M_2}{3} \left(\frac{\Delta x}{2}\right)^3$$

$$= n \frac{M_2}{3} \left(\frac{b-a}{2n}\right)^3$$

$$= \frac{M_2(b-a)^3}{24n^2}$$

## Appendix 3: Riemann Sum Worksheets

- Find left and right Riemann sum approximations to  $\int_0^{\pi} \sin(x^2) dx$  with  $n = 5$  equal subintervals.
- Find a left Riemann sum approximation to  $\int_1^2 \frac{1}{x} dx$  with  $n = 10$  equal subintervals. Is this an over approximation or an under approximation?
- Find a right Riemann sum approximation to  $\int_1^2 \frac{1}{x} dx$  with  $n = 10$  equal subintervals. Is this an over approximation or an under approximation?
- Using the approximations found in 2 and 3, find upper and lower bounds for  $\int_1^2 \frac{1}{x} dx$ .
- The following table gives data from a two-hour trip. Use the table to get both left and right Riemann sum approximations for the total distance traveled.

Time (minutes)	0	10	20	30	40	50	60	70	80	90	100	110	120
Speed (mph)	0	5	15	30	50	55	60	57	46	35	30	27	0

- Approximate  $\pi$  using both left and right Riemann sum approximations to  $\int_{-1}^1 2\sqrt{1-x^2} dx$ . Use any number of subintervals you desire.
- Use the partition  $P : 0 < 1.1 < 2.2 < 3.0 < 4.5 < 5.8 < 7.5 < 10$  of the interval  $[0, 10]$  to find left and right Riemann sum approximations for  $\int_0^{10} \sqrt{x^3 + 1} dx$ .

### Midpoint Riemann Sums

- Find the midpoint Riemann sum approximation to  $\int_0^{\pi} \sin(x^2) dx$  with  $n = 7$  equal subintervals.
- Find a midpoint Riemann sum approximation to  $\int_1^2 \frac{1}{x} dx$  with  $n = 10$  equal subintervals. Find the midpoint Riemann sum approximation to the integral with  $n = 20$  equal subintervals.
- The following table gives data from a two-hour trip. Use the table to get a midpoint Riemann sum approximation for the total distance traveled. [Hint: Use 20-minute subintervals.]

Time (minutes)	0	10	20	30	40	50	60	70	80	90	100	110	120
Speed (mph)	0	5	15	30	50	55	60	57	46	35	30	27	0

- Approximate  $\pi$  using a midpoint Riemann sum approximation to  $\int_{-1}^1 2\sqrt{1-x^2} dx$ . Use any number of subintervals you desire.

5. Using the partition  $P : 0 < 1.1 < 2.2 < 3.0 < 4.5 < 5.8 < 7.5 < 10$  of the interval  $[0, 10]$  to find a midpoint Riemann sum approximation for  $\int_0^{10} \sqrt{x^3 + 1} dx$ .

### Error in Estimating Integrals

1. Find the maximum error made in approximating  $\int_0^{\pi} \sin(x^2) dx$  with either a left or right Riemann sum when  $n = 10$  equal subintervals. How many subintervals would need to be used to guarantee an error  $E$  with  $|E| < 0.00005$ ?
2. Find the maximum error made in approximating  $\int_0^{\pi} \sin(x^2) dx$  with a midpoint Riemann sum when  $n = 10$  equal subintervals. How many subintervals would need to be used to guarantee an error  $E$  with  $|E| < 0.00005$ ?
3. What is the maximum error that could be made in approximating  $\ln(2) = \int_1^2 \frac{1}{x} dx$  with a left or right Riemann sum using  $n = 15$  subintervals? How many subintervals would be needed in order to guarantee an error  $E$  with  $|E| < 0.00005$ ?
4. What is the maximum error that could be made in approximating  $\ln(2) = \int_1^2 \frac{1}{x} dx$  with a midpoint Riemann sum using  $n = 15$  subintervals? How many subintervals would be needed in order to guarantee an error  $E$  with  $|E| < 0.00005$ ?
5. What is the maximum error that could be made in approximating  $\int_0^{10} \sqrt{x^3 + 1} dx$  with a left or right Riemann sum using  $n = 20$  subintervals? How many subintervals would be needed in order to guarantee an error  $E$  with  $|E| < 0.00005$ ?
6. What is the maximum error that could be made in approximating  $\int_0^{10} \sqrt{x^3 + 1} dx$  with a midpoint Riemann sum using  $n = 20$  subintervals? How many subintervals would be needed in order to guarantee an error  $E$  with  $|E| < 0.00005$ ?
7. What is the maximum error that could be made in approximating  $\pi = \int_{-1}^1 2\sqrt{1 - x^2} dx$  with a right or left Riemann sum with  $n = 10$  subintervals? What is the problem you encounter in finding this maximum error? Do you have any suggestions to remedy this problem?

## Appendix 4: Solutions to Riemann Sum Worksheets

### Left and Right Riemann Sum Worksheet Solutions

1. The partition of the interval will be  $P: 0 < \frac{\pi}{5} < \frac{2\pi}{5} < \frac{3\pi}{5} < \frac{4\pi}{5} < \pi$ . The left sum approximation of the integral will be

$$\sin(0) \cdot \frac{\pi}{5} + \sin\left(\left(\frac{\pi}{5}\right)^2\right) \cdot \frac{\pi}{5} + \sin\left(\left(\frac{2\pi}{5}\right)^2\right) \cdot \frac{\pi}{5} + \sin\left(\left(\frac{3\pi}{5}\right)^2\right) \cdot \frac{\pi}{5} + \sin\left(\left(\frac{4\pi}{5}\right)^2\right) \cdot \frac{\pi}{5}.$$

Evaluating each term using a calculator, we get  $0 + 0.2416569462 + 0.6282966774 + (-0.2512975162) + 0.02095776796 + 0.63961387536$ . A right sum approximation of the integral would be

$$\sin\left(\left(\frac{\pi}{5}\right)^2\right) \cdot \frac{\pi}{5} + \sin\left(\left(\frac{2\pi}{5}\right)^2\right) \cdot \frac{\pi}{5} + \sin\left(\left(\frac{3\pi}{5}\right)^2\right) \cdot \frac{\pi}{5} + \sin\left(\left(\frac{4\pi}{5}\right)^2\right) \cdot \frac{\pi}{5} + \sin(\pi^2) \cdot \frac{\pi}{5},$$

which gives  $0.2416569462 + 0.6282966774 + (-0.2512975162) + 0.02095776796 + (-0.2703662302)$ , which gives  $0.36924764516$ .

2. The partition of the interval is  $P: 1 < 1.1 < 1.2 < 1.3 < 1.4 < 1.5 < 1.6 < 1.7 < 1.8 < 1.9 < 2$ . The left sum approximation of the integral is

$$\frac{1}{1} \cdot \frac{1}{10} + \frac{1}{1.1} \cdot \frac{1}{10} + \frac{1}{1.2} \cdot \frac{1}{10} + \frac{1}{1.3} \cdot \frac{1}{10} + \frac{1}{1.4} \cdot \frac{1}{10} + \frac{1}{1.5} \cdot \frac{1}{10} + \frac{1}{1.6} \cdot \frac{1}{10} + \frac{1}{1.7} \cdot \frac{1}{10} + \frac{1}{1.8} \cdot \frac{1}{10} + \frac{1}{1.9} \cdot \frac{1}{10}.$$

Evaluating term by term, we get

$$0.1 + 0.09090909091 + 0.08333333333 + 0.07692307692 + 0.07142857143 + 0.06666666667 + 0.06250000000 + 0.05882352941 + 0.05555555556 + 0.05263157895 \approx 0.7187714032.$$

This is an overapproximation because  $f(x) = \frac{1}{x}$  is decreasing on the interval. Note that geometrically this means the rectangles determined by using the left endpoints of the subintervals will be circumscribed.

3. The right sum approximation of the integral is

$$\frac{1}{1.1} \cdot \frac{1}{10} + \frac{1}{1.2} \cdot \frac{1}{10} + \frac{1}{1.3} \cdot \frac{1}{10} + \frac{1}{1.4} \cdot \frac{1}{10} + \frac{1}{1.5} \cdot \frac{1}{10} + \frac{1}{1.6} \cdot \frac{1}{10} + \frac{1}{1.7} \cdot \frac{1}{10} + \frac{1}{1.8} \cdot \frac{1}{10} + \frac{1}{1.9} \cdot \frac{1}{10} + \frac{1}{2} \cdot \frac{1}{10}$$

Evaluating this term by term, we get

$$0.09090909091 + 0.08333333333 + 0.07692307692 + 0.07142857143 + 0.06666666667 + 0.06250000000 + 0.05882352941 + 0.05555555556 + 0.05263157895 + 0.05 \approx 0.6687714032.$$

**Special Focus: Approximation**

This will be an underapproximation because  $f(x) = \frac{1}{x}$  is decreasing and nonnegative on the interval. Note that geometrically this means the rectangles determined by using the right endpoints of the subintervals will be inscribed.

4. The estimates obtained by problems 3 and 4 imply

$$0.6687714032 < \int_1^2 \frac{1}{x} dx < 0.7187714032.$$

5. We first note that all subintervals have a width of 10 min. = 1/6 hr. We will use 1/6 hr since the speeds are in mph. A left sum will be

$$0 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 15 \cdot \frac{1}{6} + 30 \cdot \frac{1}{6} + 50 \cdot \frac{1}{6} + 55 \cdot \frac{1}{6} + 60 \cdot \frac{1}{6} + 57 \cdot \frac{1}{6} + 46 \cdot \frac{1}{6} + 35 \cdot \frac{1}{6} + 30 \cdot \frac{1}{6} + 27 \cdot \frac{1}{6} = \frac{410}{6}.$$

The total distance traveled was about 68.3 miles. A right sum approximation will be

$$5 \cdot \frac{1}{6} + 15 \cdot \frac{1}{6} + 30 \cdot \frac{1}{6} + 50 \cdot \frac{1}{6} + 55 \cdot \frac{1}{6} + 60 \cdot \frac{1}{6} + 57 \cdot \frac{1}{6} + 46 \cdot \frac{1}{6} + 35 \cdot \frac{1}{6} + 30 \cdot \frac{1}{6} + 27 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} = \frac{410}{6},$$

which gives the same estimate as the left sum estimate.

6. Since students get to make their own choice of partition, all answers cannot be anticipated. However, here are the results for some common choices they might make with subintervals of equal width. One should note that all left and right Riemann sums for the same number of subintervals will be the same because the integrand is 0 at both endpoints of the interval  $[-1, 1]$ .

Number of subintervals	Left Riemann Sum = Right Riemann Sum
4	2.732050808
5	2.847673436
10	3.037048829
15	3.084570902
20	3.104518328
25	3.115048238
30	3.121391413
50	3.132196311
75	3.136475846
100	3.138268510
200	3.140417036
500	3.141295187
1000	3.141487472

7. A left Riemann sum here for  $g(x) = \sqrt{x^3 + 1}$  is given by

$$g(0) \cdot 1.1 + g(1.1) \cdot 1.1 + g(2.2) \cdot 0.8 + g(3.0) \cdot 1.5 + g(4.5) \cdot 1.3 + g(5.8) \cdot 1.7 + g(7.5) \cdot 2.5 \approx 101.1412655.$$

A right Riemann sum is given by

$$g(1.1) \cdot 1.1 + g(2.2) \cdot 1.1 + g(3.0) \cdot 0.8 + g(4.5) \cdot 1.5 + g(5.8) \cdot 1.3 + g(7.5) \cdot 1.7 + g(10) \cdot 2.5 \approx 156.3244446$$

### Midpoint Riemann Sums Worksheet Solutions

1. The partition of the interval will be  $P: 0 < \frac{\pi}{7} < \frac{2\pi}{7} < \frac{3\pi}{7} < \frac{4\pi}{7} < \frac{5\pi}{7} < \frac{6\pi}{7} < \pi$ .

The midpoint sum approximation of the integral will be

$$\sin\left(\left(\frac{\pi}{14}\right)^2\right) \cdot \frac{\pi}{7} + \sin\left(\left(\frac{3\pi}{14}\right)^2\right) \cdot \frac{\pi}{7} + \sin\left(\left(\frac{5\pi}{14}\right)^2\right) \cdot \frac{\pi}{7} + \sin\left(\left(\frac{7\pi}{14}\right)^2\right) \cdot \frac{\pi}{7} + \sin\left(\left(\frac{9\pi}{14}\right)^2\right) \cdot \frac{\pi}{7} + \sin\left(\left(\frac{11\pi}{14}\right)^2\right) \cdot \frac{\pi}{7} + \sin\left(\left(\frac{13\pi}{14}\right)^2\right) \cdot \frac{\pi}{7}$$

Evaluating each term using a calculator, we get

$$0.02258977764 + 0.1965026913 + 0.4271428981 + 0.2801699042 + (-0.3616813335) + (-0.08485453354) + 0.3556360905 \approx 0.8355054947.$$

2. The partition of the interval is  $P: 1 < 1.1 < 1.2 < 1.2 < 1.4 < 1.5 < 1.6 < 1.7 < 1.8 < 1.9 < 2$ . The midpoint sum approximation of the integral is

$$\frac{1}{1.05} \cdot \frac{1}{10} + \frac{1}{1.15} \cdot \frac{1}{10} + \frac{1}{1.25} \cdot \frac{1}{10} + \frac{1}{1.35} \cdot \frac{1}{10} + \frac{1}{1.45} \cdot \frac{1}{10} + \frac{1}{1.55} \cdot \frac{1}{10} + \frac{1}{1.65} \cdot \frac{1}{10} + \frac{1}{1.75} \cdot \frac{1}{10} + \frac{1}{1.85} \cdot \frac{1}{10} + \frac{1}{1.95} \cdot \frac{1}{10}$$

Evaluating term, by term we get

$$0.09523809524 + 0.08695652174 + 0.08 + 0.07407407407 + 0.06896551724 + 0.06451612903 + 0.06060606061 + 0.05714285714 + 0.05405405405 + 0.05128205128 \approx 0.6928353603.$$

3. We first note that all subintervals have a width of 10 min. =  $\frac{1}{6}$  hr. However, to get a midpoint Riemann sum based upon the data there, we must use subintervals of width 20 min. =  $\frac{1}{3}$  hr. The midpoint Riemann sum will be

$$5 \cdot \frac{1}{3} + 30 \cdot \frac{1}{3} + 55 \cdot \frac{1}{3} + 57 \cdot \frac{1}{3} + 35 \cdot \frac{1}{3} + 27 \cdot \frac{1}{3} = \frac{209}{3} \approx 69.7 \text{ mi.}$$

4. Because students get to make their own choice of partition, all answers cannot be anticipated. However, here are the results for some common choices they might make with subintervals of equal width. One should note that all left and right Riemann sums for the same number of subintervals will be the same because the integrand is 0 at both endpoints of the interval  $[-1, 1]$ .

Number of subintervals	Left Riemann Sum = Right Riemann Sum
4	3.259367329
5	3.226424222
10	3.171987824
15	3.158211924
20	3.152411434
25	3.149344383
30	3.147494899
50	3.144340715
75	3.143089854
100	3.142565554
200	3.141936857
500	3.141679766
1000	3.141623447

7. A midpoint Riemann sum here for  $g(x) = \sqrt{x^3 + 1}$  is given by  
 $g(0.55) \cdot 1.1 + g(1.65) \cdot 1.1 + g(2.6) \cdot 0.8 + g(3.75) \cdot 1.5 + g(5.15) \cdot 1.3$   
 $+ g(6.65) \cdot 1.7 + g(8.75) \cdot 2.5 \approx 180.6880110$

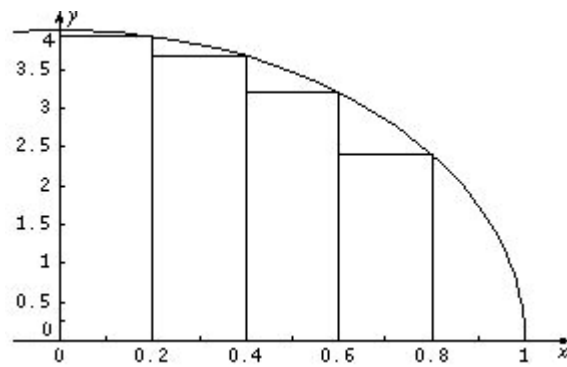
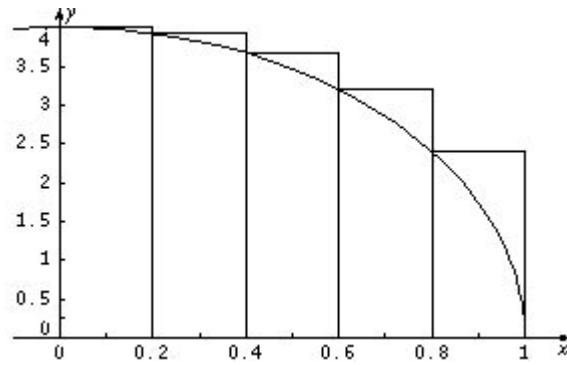
**Error in Estimating Integrals Worksheet Solutions**

- The formula for the maximum error  $E$  is given by  $|E| \leq \frac{M_1(b-a)^2}{2n}$ . For  $f(x) = \sin(x^2)$  we know that  $f'(x) = 2x\cos(x^2)$ . Graphing  $y = |f'(x)|$  we find that  $M_1 \leq 6.2$ . Thus we have  $|E| \leq \frac{6.2(\pi-0)^2}{2 \cdot 10} \approx 3.0595774$ .
- The formula for the maximum error  $E$  is given by  $|E| \leq \frac{M_2(b-a)^3}{24n^2}$ . For  $f(x) = \sin(x^2)$  we know that  $f'(x) = 2x\cos(x^2)$  and  $f''(x) = 2\cos(x^2) - 4x^2 \sin(x^2)$ . Graphing  $y = |f''(x)|$  we find that  $M_2 \leq 32$ . Thus we have  $|E| \leq \frac{32(\pi-0)^3}{24 \cdot 100} \approx 0.4134170224$ .
- We know that  $(\frac{1}{x})' = -\frac{1}{x^2}$ . On the interval  $[1, 2]$  the absolute value of  $-\frac{1}{x^2}$  is maximized when  $x = 1$ . Thus  $M_1 = 1$ , and using the formula  $|E| \leq \frac{M_1(b-a)^2}{2n}$  we find  $|E| \leq \frac{1 \cdot (2-1)^2}{2 \cdot 15} \approx 0.0333333333$ . To guarantee  $|E| < 0.00005$  we must have  $\frac{M_1(b-a)^2}{2n} < 0.00005$  or  $n > \frac{1 \cdot (2-1)^2}{2(0.00005)} = \frac{1}{0.0001} = 10,000$ . Thus we must use more than 10,000 equal subintervals.



4. We know that  $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$  and equal  $\left(-\frac{1}{x^2}\right)' = \frac{2}{x^3}$ . On the interval  $[1, 2]$  the absolute value of  $\frac{2}{x^3}$  is maximized when  $x = 1$ . Thus  $M_2 = 2$ , and using the formula  $|E| \leq \frac{M_2(b-a)^3}{24n^2}$  we find  $|E| \leq \frac{2 \cdot (2-1)^3}{24 \cdot 15^2} < 0.0003704$ . To guarantee  $|E| < 0.00005$  we must have  $\frac{M_2(b-a)^3}{24n^2} < 0.00005$  or  $n^2 > \frac{2 \cdot (2-1)^3}{24(0.00005)} = \frac{5000}{3}$ . This means we must have  $n > \sqrt{\frac{5000}{3}} \approx 40.825$ . Thus we must use 41 or more subintervals.
5. Maximum error  $E$  satisfies  $|E| \leq \frac{M_1(b-a)^2}{2n}$ . For  $f(x) = \sqrt{x^3 + 1}$ , we know that  $f'(x) = \frac{3x^2}{2\sqrt{x^3 + 1}}$ . Graphing  $y = |f'(x)|$  we see that its maximum value on  $[0, 10]$  occurs when  $x = 10$ . Computing  $f'(10)$  we find that  $M_1 \leq 4.74105$ . Thus we have  $|E| \leq \frac{4.74105(10-0)^2}{2 \cdot 20} \approx 11.85262500$ . To guarantee  $|E| < 0.00005$  we must have  $\frac{M_1(b-a)^2}{2n} < 0.00005$  or  $n > \frac{4.74105 \cdot (10-0)^2}{2(0.00005)} = \frac{474.105}{0.0001} = 4,74,050$ . Thus we must use more than 4,741,050 equal subintervals.
6. For  $f(x) = \sqrt{x^3 + 1}$  we know that  $f'(x) = \frac{3x^2}{2\sqrt{x^3 + 1}}$  and  $f''(x) = \frac{3x(x^3 + 4)}{4(\sqrt{x^3 + 1})^3}$ . Graphing  $y = |f''(x)|$  on the interval  $[0, 10]$  gives  $M_2 \leq 1.5$ , and using the formula  $|E| \leq \frac{M_2(b-a)^3}{24n^2}$  we find  $|E| \leq \frac{1.5 \cdot (10-0)^3}{24 \cdot 20^2} = 0.15625$ . To guarantee  $|E| < 0.00005$  we must have  $\frac{M_2(b-a)^3}{24n^2} < 0.00005$  or  $n^2 > \frac{1.5 \cdot (10-0)^3}{24(0.00005)} = 1,250,000$ . This means we must have  $n > \sqrt{1,250,000} \approx 1118.034$ . Thus we must use 1,119 or more subintervals.
7. For  $f(x) = 2\sqrt{1-x^2}$  we have  $f'(x) = \frac{-2x}{\sqrt{1-x^2}}$ . This function is unbounded on the interval  $[-1, 1]$ . Thus we cannot get an upper bound on the error using the given formulas. One could eliminate the problem at one endpoint by using  $\int_0^1 4\sqrt{1-x^2} dx$  to estimate  $\pi$ . Using 5 subintervals for this integral would be equivalent to the 10 subintervals stipulated. However, there is still a problem at the right endpoint of the interval  $[0, 1]$ . One could use geometric ideas to find the maximum error. Consider the graphs below.

## Special Focus: Approximation



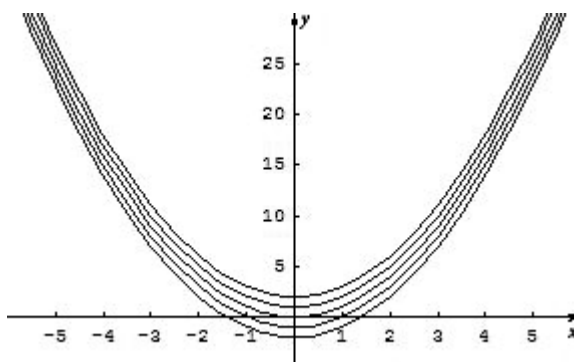
We observe that the left sum gives an overestimate and the right sum gives an underestimate. The left Riemann sum with  $n = 5$  equal subintervals, is 3.437048829, and the right Riemann sum, is 2.637048829. Thus the maximum error made is the difference of these numbers, which is 0.8.

## Approximating Solutions to Differential Equations: Slope Fields and Euler's Method

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The concept of slope fields first appeared in the AP Calculus BC 1998 Exam, and later was added to the AP Calculus AB 2004 Exam. In their simplest form, slope fields are nothing more than a graphical approximation of a family of vertically stacked functions; in other words, they are a *field of slope segments*, or a *field of tangent line segments*. Slope fields are not restricted to graphs that are mathematical functions, but it is best to introduce this topic by using relatively easy functions. Slope fields can also be thought of as the graphical approximation for the solution to a particular differential equation (DE).

When working with antiderivatives, I want my students to discover that there are an infinite number of solutions to a simple differential equation such as  $\frac{dy}{dx} = 2x$ . Students quickly see that  $F(x) = x^2$  is one of the many functions that could actually be a solution to this differential equation. In fact,  $F(x)$  can be  $x^2 + C$  and the derivative is still  $\frac{dy}{dx} = 2x$ . The constant of integration ( $+C$ ) included in the solution becomes part of the student's intuitive thinking (although many unfortunately will forget this  $+C$  on future problems/exams). The figure below shows the graphs of the following functions:  $F(x) = x^2 + 2$ ,  $F(x) = x^2 + 1$ ,  $F(x) = x^2$ ,  $F(x) = x^2 - 1$ , and  $F(x) = x^2 - 2$ .



On the first day of teaching antiderivatives I introduce slope fields (sometimes called direction fields) to my students. From my previous differential equation example,  $\frac{dy}{dx} = 2x$ , we graph a *family* of vertically stacked functions on the whiteboard to represent the potential solutions for this DE. From this graph I develop a slope field by making tangent-line segments that approximate the graphical solution to the DE. It is easy for my students to discover the parabolic solution  $F(x) = x^2 + C$  to the DE embedded in the slope-field graph.

**Special Focus: Approximation**

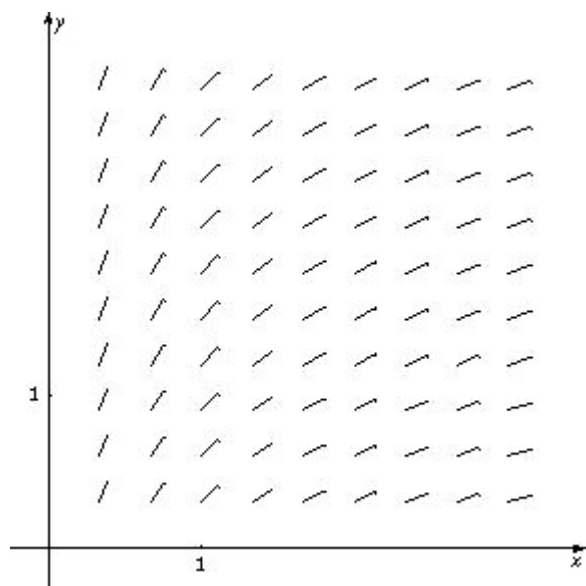
To produce a slope field for the differential equation  $\frac{dy}{dx} = 2x$ , students need to calculate the slope at various coordinate points. At every point where  $x = 0$  (regardless of the  $y$ -value) the slope is equal to zero (0), and students should draw a short horizontal line segment at these points. At every point where  $x = 1$  (once again, regardless of the  $y$ -value) the slope is equal to two (2), and students should draw a short line segment with an approximate slope of two (2). After students have drawn these tangent-line segments, they erase the *original* family of parabolic curves. Students quickly discover how easy it is to construct a slope field, give a specific differential equation.

Consider the following problem from the AP Course Description.

**AP Calculus Course Description May 2006, May 2007**

**BC Multiple-Choice Problem #7, Noncalculator**

In this multiple-choice question, students are given a slope field in Quadrant I and asked which function could be a particular solution. If students are familiar with the graphs of these basic functions, then answer E is the obvious choice because one can see the logarithmic curve embedded in the slope field.



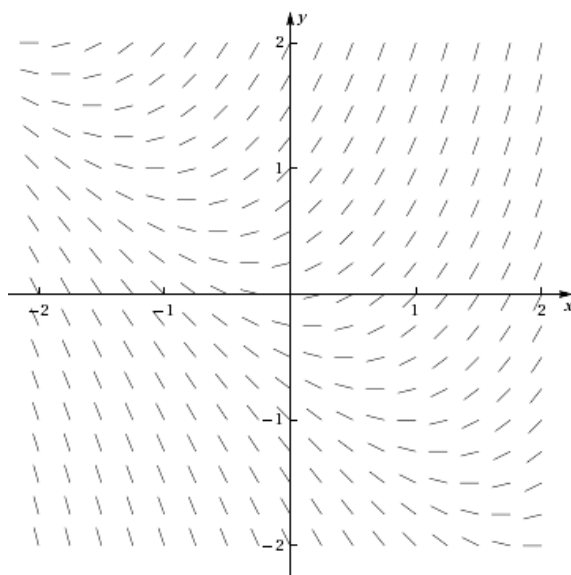
The slope field for a certain differential equation is shown above. Which of the following could be a specific solution to that differential equation?

- (A)  $y = x^2$
- (B)  $y = e^x$
- (C)  $y = e^{-x}$
- (D)  $y = \cos x$
- (E)  $y = \ln x$

When constructing slope fields we select certain points on the graph and use the DE to find the slope of the function at that point. We then draw a short line segment with that slope on our slope field graph. By calculating the different slopes using a number of different coordinate points, students can approximate the graph of the (solution) function using a small number of points and line segments that are in the domain of the function. Students can draw a particular solution on the slope-field graph that they can see embedded in the slope field. This should help the students visualize what a potential solution to the differential equation could look like.

Students should also be able to match slope fields with their differential equations without actually solving the differential equation or sketching the solution curves. Consider the following example.

### 1998 BC Multiple Choice #24, Noncalculator



24. Shown above is a slope field for which of the following differential equations?

(A)  $\frac{dy}{dx} = 1 + x$     (B)  $\frac{dy}{dx} = x^2$     (C)  $\frac{dy}{dx} = x + y$     (D)  $\frac{dy}{dx} = \frac{x}{y}$     (E)  $\frac{dy}{dx} = \ln x$

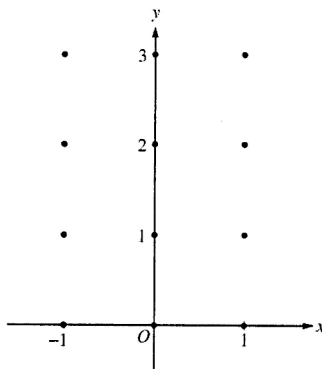
In this problem students could find the family of solution functions by solving each differential equation. However, this would be time consuming and is not possible for part (C). If students can see that the slope of the graph is zero at the point  $(-2, 2)$  (or at numerous other points and at every point  $(x, -x)$ ), by the process of elimination answer (C) is the obvious choice.

### 2004 AB Free-Response #5, Form B

Students have been asked to draw the slope field using a differential equation and a select number of points on recent free-response questions. If students have solved this type of problem before, Part A (in the problem below) should now be relatively easy for them to complete. The AP Calculus Readers are looking for patterns in slopes (positive, negative, zero, increasing or decreasing in value); students can approximate these segments without any measurements or rulers.

5. Consider the differential equation  $\frac{dy}{dx} = x^2(y - 1)$ .

- (a) On the axes provided, sketch a slope field for the given differential equation at the 12 points indicated. (**Note: Use the axes provided in the pink test booklet.**)



**(I have not included parts (b) and (c) of this free-response problem.)**

A slope field can be thought of as an approximation to the graphical solution to a differential equation; it is one component of the Rule of Four, which calls for describing solutions graphically, numerically, analytically, and verbally. Students in Calculus AB are required to solve differential equations analytically by using techniques that include antidifferentiation and the separation of variables. Since I introduce slope fields so early in

the course, students should feel comfortable using both graphical and analytical methods to solve a number of different types of differential equations. I also teach Euler's Method in my AB classes (even though this is a BC topic) and want my students to see this process as a numerical approach for approximating a particular solution to a given differential equation (another component of the Rule of Four).

### Euler's Method (BC Topic)

Leonhard Euler lived from 1707 to 1783 and made numerous contributions to mathematics, acoustics, astronomy, magnetism, and fluid mechanics. Euler introduced symbols for function notation  $f(x)$ ,  $\pi$ , the natural base  $e$ , and  $i = \sqrt{-1}$ , as well as Euler's Method. This technique is used to approximate a particular solution to a given differential equation when we have:

- A differential equation (this is difficult to integrate within the scope of AB/BC Calculus course curriculums);
- An initial point on our solution curve;
- An ending  $x$  value; or
- A specific step-size or increment which may help to determine our ending  $x$  value.

By using the given initial point, we can closely follow our function by drawing a number of short line segments and connecting them to help approximate our solution curve. Our final point (and in fact any point) will be an approximation of the corresponding (same  $x$ -value) point on our solution curve.

### 2003 BC Multiple Choice #5

5. Let  $y = f(x)$  be the solution to the differential equation  $\frac{dy}{dx} = x + y$  with initial condition  $f(1) = 2$ . What is the approximation for  $f(2)$  if Euler's method is used, starting at  $x = 1$  with a step size of 0.5?

- (A) 3      (B) 5      (C) 6      (D) 10      (E) 12

This differential equation is not easy to solve, even for BC students. We know the DE and a point on our function ( $f(1) = 2$ ), so by using Euler's Method we can approximate  $f(2)$ . By substituting the point  $(1, 2)$  into our DE we get a slope of three (3). Our step size  $\Delta x$  is equal to 0.5, so we will have two steps (line segments) from our original point to approximate  $f(2)$ . By adding our constant  $\Delta x$  value (0.5) and our  $\Delta y$  values to our starting point  $(1, 2)$ , we can approximate that the point  $(2, 6)$  is *close* to the actual point on our function/solution curve, and the correct answer is (C).

**Special Focus: Approximation**

$x$	$y$ (approximate)	$y'$ (Slope)	$\Delta x$	$\Delta y = y' \Delta x$
1	2	3	.5	1.5
$1.5(1 + 0.5)$	$3.5(2 + 1.5)$	5	.5	2.5
$2(1.5 + 0.5)$	$6(3.5 + 2.5)$			

Students can then be led to discover that our approximation for  $f(2)$  would be more accurate if we were to draw more line segments based upon a smaller step size. As the number of steps approaches infinity, our estimated value will approach the actual value for  $f(2)$ . There are a number of excellent graphing calculator programs that make this process easier and faster, and allow students to grasp this concept more effectively.

Both slope fields and Euler's Method use tangent lines to give an approximation for a function, either graphically or numerically. The concept of local linearity, of which students should have a solid grasp at this stage of the course, helps them to better understand these two approximation techniques.



## Approximation Using Euler's Method

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Imagine trying to measure a curve using a rigid ruler. Because the contours of the curve cannot be followed with the ruler, one would have to make small interval measurements to form an estimate. This is similar to a process of estimation called *Euler's Method*.

Let  $\frac{dy}{dx} = f(x, y)$  (that is, a differential equation in  $x$  and/or  $y$ ), and let  $(x_0, y_0)$  be an initial condition of its solution. Further, let's suppose that the differential equation cannot be solved using the known method of separating variables—or, for that matter, it could be unsolvable by *any* method. Then, we can only estimate points on the solution curve given the information provided. One of the simplest approaches to this kind of estimated solution relies on the approximation formula  $\Delta y \approx \frac{dy}{dx} \cdot \Delta x$  and is known as *Euler's Method*. In the example below, we will use a differential equation that can be solved by separation of variables. One can then readily compare the solution estimates provided by Euler's Method with the actual values of the particular solution.

*Illustration:*  $\frac{dy}{dx} = xy$  and  $(x_0, y_0) = (0, 1)$

From our work with differentials we know that for a differentiable function with  $\Delta x$  sufficiently small,  $\Delta y \approx \frac{dy}{dx} \cdot \Delta x$ , which is to say that the curve is approximated by its tangent line over a small interval. Extending this reasoning, we can create a recursion formula to develop successive tangent-line approximations. The formula can be expressed as  $\Delta y \approx \frac{dy}{dx} \cdot \Delta x$ , each time adding the value of  $\Delta y$  to the previous  $y$  value to generate the next point. Suppose the values of  $x$  increase by 0.1 at each step.

Let  $\Delta x = 0.1$ ; since  $\frac{dy}{dx} = xy$ , then at  $(0, 1)$  we have  $\frac{dy}{dx} = 0$  and  $\Delta y \approx 0(.1) = 0$ . This brings us to an estimate of a nearby point on the solution curve,  $(0.1, 1)$ . Call this point  $(x_1, y_1)$ .

Now we can repeat this process. At  $(.1, 1)$ ,  $\frac{dy}{dx} = .1$  and  $\Delta y \approx .1(.1) = .01$ .

Thus, our next point is  $(x_2, y_2) = (.2, 1.01)$ .

As we continue, the  $y$ -estimates become more complicated to calculate. It might help to use the recursive formula:  $y_{n+1} \approx y_n + f'(x_n) \cdot \Delta x$ , which computes each new  $\Delta y$  and adds it to the predecessor. Here is a closer look at the individual steps in this process:

**Special Focus: Approximation**

$$f(x_{n+1}) \approx f(x_n) + f'(x_n) \cdot \Delta x$$

$$f(.1) \approx f(0) + f'(0) \cdot (.1)$$

$$= 1 + 0 \cdot (.1) = 1$$

$$f(.2) \approx f(.1) + f'(.1) \cdot (.1)$$

$$= 1 + (.1)(.1) = 1.01$$

$$f(.3) \approx f(.2) + f'(.2) \cdot (.1)$$

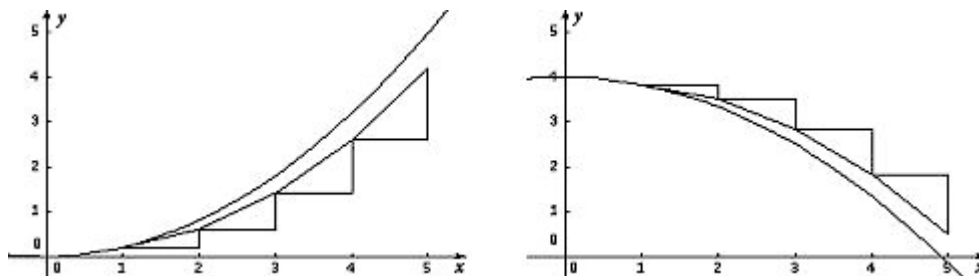
$$= 1.01 + .0202 = 1.0302$$

Below is a table that compares the values of these estimates with the actual values of the solution curve,  $y = e^{-5x^2}$ .

$x$	0	0.1	0.2	0.3	0.4	0.5
Euler $y$	1	1	1.01	1.0302	1.0611	1.1036
$y = e^{-5x^2}$	1	1.005	1.020	1.046	1.083	1.133

For a value of  $\Delta x$  that is not exceptionally small, the method provides estimates that are remarkably close.

Whether Euler’s Method underestimates or overestimates the actual function values is largely dependent on concavity. The graphs of the functions and the associated Euler tangents shown below suggest this relationship. These figures also suggest that as we move further from the initial point, the error propagated is greater and greater.



When a portion of the curve is concave up, the approximations given by Euler’s Method will underestimate the function values; when the graph is concave down, the approximations will overestimate the function values. As usual, when we make the increments of  $x$  (or steps) smaller, we improve the estimate accordingly. This fact is demonstrated and amplified by a surprising observation below.

## A Postscript on Euler's Method

Euler's Method has a well-founded and rather unexpected connection to a familiar observation about approximations. Consider the following example.

Given a continuous function  $f'$ , find an approximation to  $f$  with the initial condition  $(x_0, y_0)$  and evaluate it on the interval  $[x_0, x_n]$ , where  $x_n = n \cdot \Delta x$ .

Solution: For small  $\Delta x$ ,  $\Delta y \approx f'(x_0) \cdot \Delta x$ . It follows that

$$f(x_1) \approx f(x_0) + f'(x_0) \cdot \Delta x \quad (1)$$

$$\text{And, } f(x_2) \approx f(x_1) + f'(x_1) \cdot \Delta x \quad (2)$$

$$= f(x_0) + f'(x_0) \cdot \Delta x + f'(x_1) \cdot \Delta x \quad (3)$$

Continuing in this way, we eventually obtain:

$$f(x_n) \approx f(x_0) + f'(x_0) \cdot \Delta x + f'(x_1) \cdot \Delta x + \cdots + f'(x_{n-1}) \cdot \Delta x \quad (4)$$

Now, using a left Riemann sum, the area bounded by  $f'$  from  $x_0$  to  $x_n$  can be approximated as:

$$A \approx f'(x_0) \cdot \Delta x + f'(x_1) \cdot \Delta x + \cdots + f'(x_{n-1}) \cdot \Delta x \quad (5)$$

$$\text{But, } f(x_n) \approx f(x_0) + f'(x_0) \cdot \Delta x + f'(x_1) \cdot \Delta x + \cdots + f'(x_{n-1}) \cdot \Delta x \quad (6)$$

$$\text{Or, } f(x_n) - f(x_0) \approx f'(x_0) \cdot \Delta x + f'(x_1) \cdot \Delta x + \cdots + f'(x_{n-1}) \cdot \Delta x \quad (7)$$

As  $\Delta x$  approaches 0, the right-hand side of line (7) becomes  $A$  and is equal to the left-hand side. That is,

$$A = f(x_n) - f(x_0)$$

This illustrates the evaluation part of the Fundamental Theorem of Calculus; that is,

$$\int_{x_0}^{x_n} f'(x) dx = f(x_n) - f(x_0)$$

## Using the Derivative to Approximate Function Values

In the explorations in the article that follows, we will investigate in more detail tangent-line approximations and polynomial approximations. Teachers of AP Calculus AB could use just the tangent line part of the activity (though they could certainly do both parts). For AP Calculus BC teachers, the second part of the activity provides a nice motivation for Taylor polynomial approximations.

## The Tangent Line as “The Best” Linear Approximation

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A tangent line to a curve,  $y = f(x)$ , at a point where  $x = a$  has two important properties:

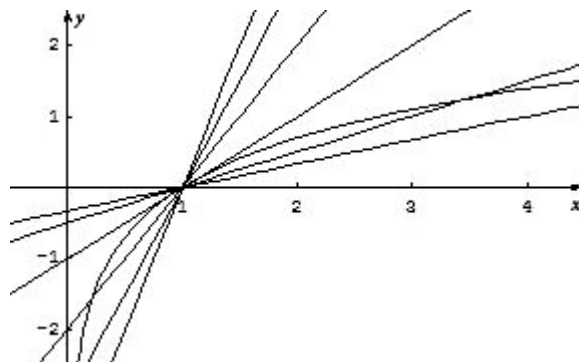
- it contains the point  $(a, f(a))$
- it has slope  $m = f'(a)$

For example, the line  $y = \frac{1}{2}x + \frac{1}{2}$  is tangent to the graph of  $f(x) = \sqrt{x}$  at  $x = 1$  since it passes through  $(1,1)$  and has the same slope,  $\frac{1}{2}$ , as  $\sqrt{x}$  has at  $x = 1$ . Near the point of tangency, the tangent line is *close* to the curve. So outputs from the line should be close to outputs from the curve. In this example, at  $x = 1.2$ , the output on the line is 1.1 and the value of  $\sqrt{1.2}$  is about 1.095445.

Higher-degree polynomials can also be used to approximate function outputs. For example, we could make the output and first *two* derivatives of a polynomial agree with those of  $\sqrt{x}$  at  $x = 1$ . A polynomial that does the trick is  $q(x) = -\frac{1}{8}x^2 + \frac{3}{4}x + \frac{3}{8}$ . Notice how much better this quadratic does in approximating  $\sqrt{1.2}$ :  $q(1.2) = 1.095$ , accurate to three decimal places. One reason polynomial approximations are important is that they can be calculated using only the four basic arithmetic operations: addition, subtraction, multiplication, and division.

### Exploration, Part 1: The Tangent Line as “The Best” Linear Approximation

In this activity, you will investigate why the tangent line is called *The Best* linear approximation to a function at a point. This activity can be done with AB or BC students. Suggested questions to ask students during the activity appear in *italics* in the narrative. In the figure below, the function  $f(x) = \ln(x)$  is graphed, along with several linear functions that intersect the graph of  $f$  at the point  $(1, 0)$ .



Any of those lines could be used to approximate the value of  $\ln(x)$  near  $x = 1$ . In this activity, we'll explore how to use linear functions to approximate function values, and why we call the tangent line *The Best*.

Now that we have inexpensive calculators to evaluate transcendental functions, it's harder to justify the use of tangent lines to approximate function values. Nonetheless, learning how to do so has more value than just to become familiar with an historic footnote. The tangent line is special for its connection with deep calculus concepts.

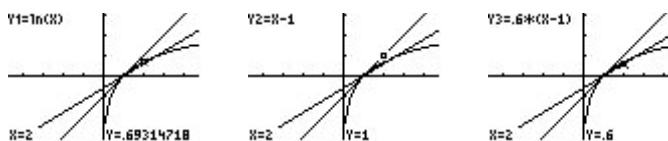
In order to use the tangent line to approximate a function, we must be able to write an equation for that line. Writing an equation for a tangent line requires that we be able to do two things: determine the exact function output at a point, and calculate the exact value of the slope at that point. Here we'll use the function  $f(x) = \ln(x)$  and the point  $(1, 0)$ . The derivative is  $f'(x) = \frac{1}{x}$  and so  $f'(1) = 1$  and our tangent line equation is  $y = x - 1$ . We'll use the calculator to investigate the tangent line and some other lines as approximations for  $\ln(x)$  near  $x = 1$ .

First, define functions as shown in the screen below, and consider their graphs in the decimal window.



What would the error be if each line were used to approximate  $\ln(1)$ ? Why?

We'll use the calculator to see how well each line does in approximating  $\ln(2)$ .



As you can see from the calculator screens above, the line through  $(1, 0)$  with the *wrong* slope actually does a much better job than the tangent line in approximating  $\ln(2)$ !

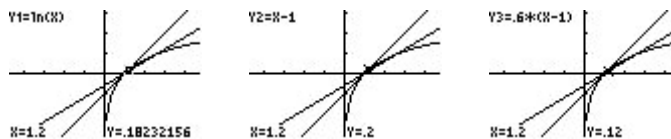
Use the second derivative to explain why the line with slope 0.6 does better at approximating  $\ln(2)$ .

From the graph, this isn't surprising. The line defined in Y3 has a slope that is less than the slope of  $\ln(x)$  at  $x = 1$ . The slope of  $\ln(x)$  is always decreasing, so it does make sense that at some point, the line with a slope that is too small will actually do better. In fact, there is even a point where that other line intersects the graph of  $y = \ln(x)$  and of course, at that point, its

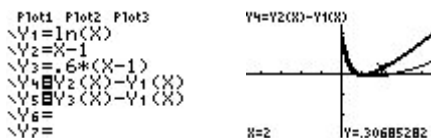
**Special Focus: Approximation**

error in approximating  $\ln(x)$  would be zero! In order to see why the tangent line is called *The Best*, we'll need to look more closely.

If you use your calculator to trace to a point that is closer to the point of tangency, say  $x = 1.2$ , you'll see that the tangent-line approximation is better.

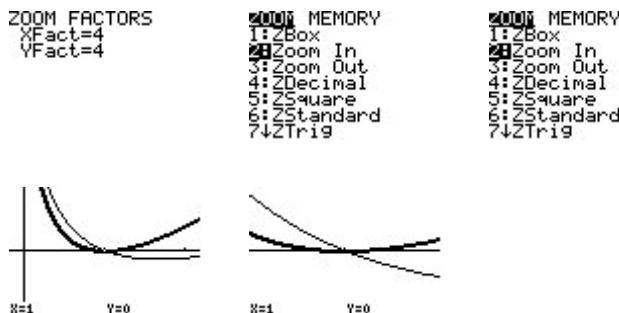


Let's define functions that reveal the *error* in each approximating line. Since our interest is focused only on the error, be sure to deselect Y1, Y2, and Y3. You might also want to change the graphing style of the tangent-line error, so you can tell it apart from the error of the line with slope 0.6.



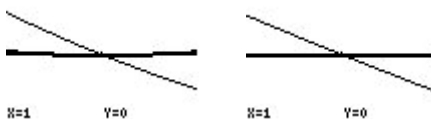
Explain why each line passes through the point (1, 0).

Notice that in this decimal view, the error from using the line with slope 0.6 seems to be closer to the x-axis, indicating that smaller error. What happens if we zoom in at the point of tangency, (1, 0)? Make sure your zoom factors are equal to start out, and be sure to position the zoom cursor on (1, 0).



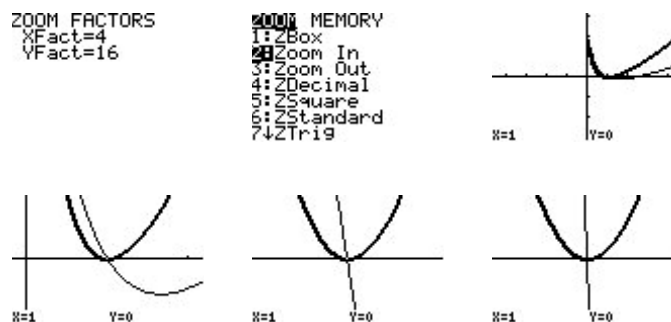
Use the second derivative to explain why the tangent-line error is always positive.

Zoom in a couple more times.



What happened to the tangent-line error as you zoomed in with equal factors?

Notice that as the values of  $x$  got closer to 1, the tangent-line error flattened out. However, the error from the other line—though it initially changed shape a bit—as we look closer continues to look linear and maintain the same steepness. Referring to the results from Appendix: Power Zooming (following the article, “Estimating Derivatives Numerically”), these features indicate that the tangent-line error is behaving with a degree greater than 1, while the other line error is behaving like a degree-one (linear) polynomial. To confirm this, go back to the decimal window, change the zoom factors to do a degree-two power zoom (here, we used XFact = 4 and YFact = 16), and try zooming in again (remember to reposition the cursor at (1, 0)).



What happened to the tangent-line error as you zoomed in with a degree-two power zoom?  
 What happened to the other line error?

Stunning! The tangent-line error keeps the same shape under a degree-two power zoom, while the other line error gets steeper. Steep error is bad!

The tangent line is special because its error behaves with a higher degree than that of any other line. Higher-degree power functions stay close to zero longer, and that’s good when the power function is measuring error.

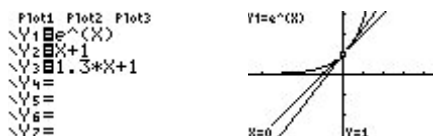
## Exploration, Part 2: The Taylor Polynomial as “The Best” Polynomial Approximation

Of course, what’s special about any tangent line is that its slope agrees with the function at the point of tangency. You could repeat this activity with any differentiable function, and the results would be the same. The error from *any* line with the wrong slope will behave like a degree one polynomial. At worst, the tangent-line error exhibits degree-two behavior. Occasionally, it’ll do even better than that. For example, if you use  $y = x$ , the tangent line to  $f(x) = \sin(x)$  at  $x = 0$ , you’ll see that the error behaves like a degree-*three* power function. This fact is intimately connected with the fact that the second derivative of  $\sin(x)$  is 0 at  $x = 0$ .

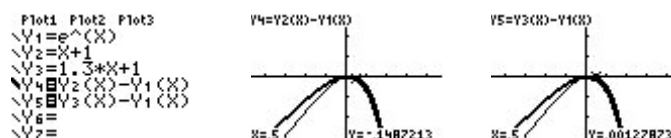
**Special Focus: Approximation**

The results of this activity lay the foundation for Taylor polynomials, a BC-only topic. Let's use a different function,  $g(x) = e^x$ , and a different point,  $(0, 1)$ , to explore further.

Since,  $g'(x) = g''(x) = e^x$ ,  $g'(0) = g''(0) = 1$ . The tangent line at  $x = 0$  is  $y = x + 1$ . We'll use another line,  $y = 1.3x + 1$ , that goes through  $(0,1)$  but has the wrong slope. Take a look at the screens below.



Now, to repeat the activity above, define the two error functions.



Trace over to  $x = 0.5$  on the two error functions, as shown above. You'll see that Y3 actually does a better job approximating  $\sqrt{e}$  than Y2 does! As before, we need to look closer to see the difference between these two linear approximations.

Use your calculator to trace to a point that is closer to the point of tangency, say  $x = 0.2$ . You'll see that the tangent-line approximation is better.



Consider a degree-two power zoom at the origin, and you should see the same results we saw earlier: The tangent-line error eventually maintains its parabolic shape, while the error in the other line gets steeper. See Appendix: Power Zooming (following the article "Estimating Derivatives Numerically").

Now we'll build a quadratic function to approximate  $e^x$  near  $x = 0$ . To do so, we'll define a quadratic whose output, slope, and second derivative all match those of  $e^x$  at  $x = 0$ . The quadratic that satisfies these requirements is  $q(x) = \frac{x^2}{2} + x + 1$ .

Verify that  $q'(0) = q''(0) = 1$ .

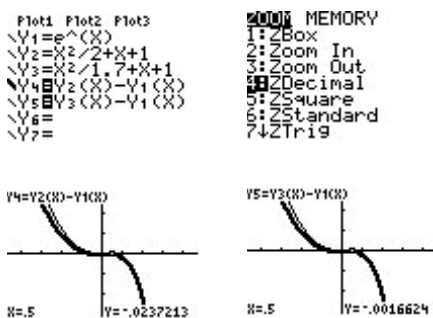
We'll also need a quadratic whose output and slope agree with  $e^x$  at  $x = 0$ , but whose second derivative is slightly off. The quadratic  $r(x) = \frac{x^2}{1.7} + x + 1$  will work.



Verify that  $r(x)$  has the correct output and slope, but that its second derivative is wrong.

Once again, we'll turn to the calculator to investigate how the error functions behave. First, though, you should take a look at Y1, Y2, and Y3 graphed together.

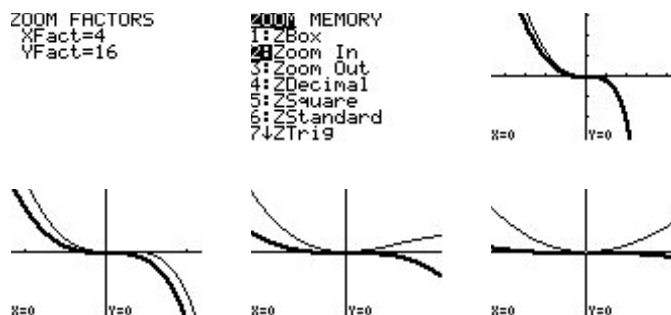
Now, check out the errors at  $x = 0.5$ . Notice that both quadratics did better than either one of the tangent lines we looked at above.



Explain why Y4 and Y5 both go through the origin.

Again, though, the *wrong* quadratic did a better job! In this case, the second derivative of Y3 is greater than 1, the second derivative of  $e^x$  at  $x = 0$ . Notice that the third derivative of  $e^x$  is also 1, and so the second derivative is increasing. So it isn't altogether surprising that Y3 did better, because its second derivative is too big. When we look closer, though, we should see the difference between Y2 and Y3.

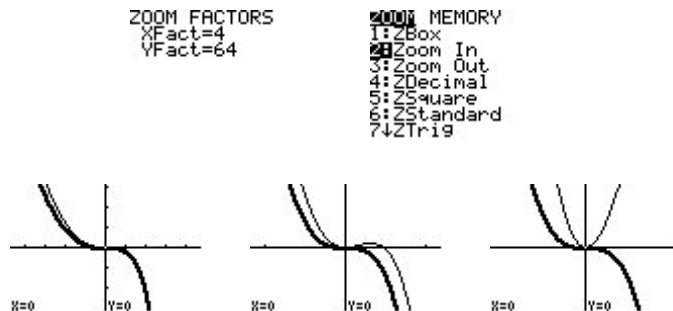
So let's try a degree-two power zoom at the origin.



The error in the *good* quadratic's error is flattening out under the degree-two power zoom, while the error in the other quadratic is maintaining the same shape.

What do the graphs tell you about the degree of the behavior of the two errors?

This indicates that the degree behavior for the good quadratic is greater than two, while the other quadratic's error is behaving like a degree-two. The natural thing to do is to try a degree-three power zoom and see whether the good quadratic's error keeps the same shape.



Pretty convincing! The other quadratic's error is getting steeper, while the good quadratic's error is maintaining its shape.

Understanding the *why* for all of this is relatively straightforward if you look at the Taylor polynomials involved. We know, for instance, that the Taylor series for  $\ln(x)$  at  $x = 1$  is  $(x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \dots$ . If we subtract this from the expression for the tangent line at  $x = 1$ ,  $x - 1$ , the lowest degree with which we are left is two.

Similarly, the McLaurin series for  $e^x$  is  $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ . Subtract our *good* quadratic,  $\frac{x^2}{2} + x + 1$ , and the lowest degree with which we are left is three. Subtract the other quadratic,  $\frac{x^2}{1.7} + x + 1$ , and we still have a degree-two term. Near where they are zero, the lower-degree terms dominate the behavior.

Note: These activities were inspired by a talk by Don Kreider at TICAP in the early 1990s.

**Reference**

Dick, Thomas P., and Charles M. Patton, *Calculus of a Single Variable*, PWS Publishing Company, Boston, 1994.

## Approximating Sums of Series and Values of Functions Series Approximations

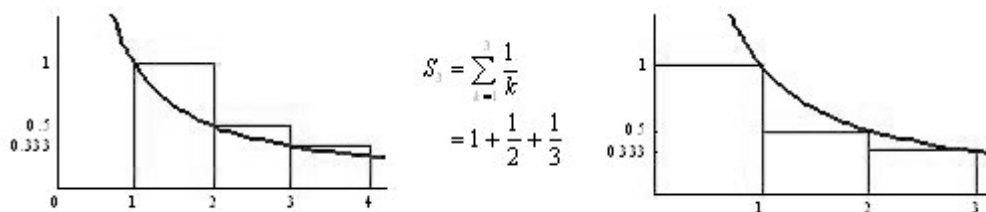
Ruth Dover  
Illinois Mathematics and Science Academy  
Aurora, Illinois

Infinite series are difficult for many students. The title itself suggests at least one reason why: The sum of an infinite series is approximated by its associated sequence of partial sums. To help students understand the sequence of partial sums, be sure to spend some time at the beginning of the study of series finding partial sums and the associated remainders in some rather simple examples. For example, consider the infinite geometric series  $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$ , where the common ratio is  $r = \frac{1}{3}$  and the first term is  $a = 1$ . In this case, we can find the infinite sum using the formula  $S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$ . We can also find some terms in the sequence of partial sums  $\{S_n\}$ . In general, for the infinite series  $\sum_{k=1}^{\infty} a_k$ , we define this sequence by letting  $S_1 = a_1$ ,  $S_2 = a_1 + a_2$ , etc., so the  $n^{\text{th}}$  term in this sequence is  $S_n = \sum_{k=1}^n a_k = a_1 + a_2 + \dots + a_n$ . For the geometric series above, the sequence of partial sums becomes  $\left\{1, \frac{4}{3}, \frac{13}{9}, \frac{40}{27}, \frac{121}{81}, \dots\right\}$ . (Be sure that you understand how to construct this sequence. Here,  $S_1 = a_1 = 1$ ,  $S_2 = a_1 + a_2 = 1 + \frac{1}{3} = \frac{4}{3}$ , etc.) Using both  $S$  and the sequence of partial sums, we may calculate the remainders  $R_n$ , the leftovers, where  $S = S_n + R_n$ , or  $R_n = S - S_n$ . For example,  $R_1 = S - S_1 = \frac{3}{2} - 1 = \frac{1}{2}$ , and  $R_2 = S - S_2 = \frac{3}{2} - \frac{4}{3} = \frac{1}{6}$ . Here, the sequence  $\{R_n\}$  becomes  $\left\{\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \frac{1}{162}, \dots\right\}$ . Working with these values and relationships can help students gain an understanding of the concepts of convergence, approximation, and error. In particular, it should be clear that as  $n$  grows larger,  $R_n$  tends toward 0.

Another interesting series that may be used to calculate some values of  $S_n$  and  $R_n$  is  $\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots$ . Using this formula, students may calculate the sequence of partial sums  $\{S_n\}$  and the remainders  $\{R_n\}$ .

Another strategy for approximating infinite sums comes from the relationship between the partial sums and associated improper integrals. Here, we'll consider the harmonic series  $\sum_{k=1}^{\infty} \frac{1}{k}$  and the integral  $\int_1^{\infty} \frac{1}{x} dx$ . To illustrate this relationship, let  $n = 3$ .

Three rectangles are drawn in each figure with areas 1, 1/2, and 1/3, corresponding to the first three terms of the series.



From these graphics, we may write the following:

$$\int_1^4 \frac{1}{x} dx \leq \sum_{k=1}^3 \frac{1}{k} \leq 1 + \int_1^3 \frac{1}{x} dx$$

If we consider a greater number of rectangles, we can generalize:

$$\int_1^{n+1} a(x) dx \leq \sum_{k=1}^n a(k) \leq a_1 + \int_1^n a(x) dx$$

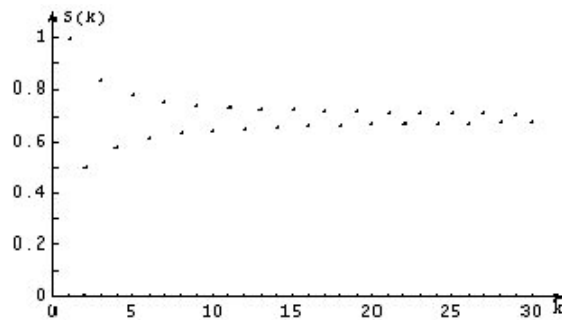
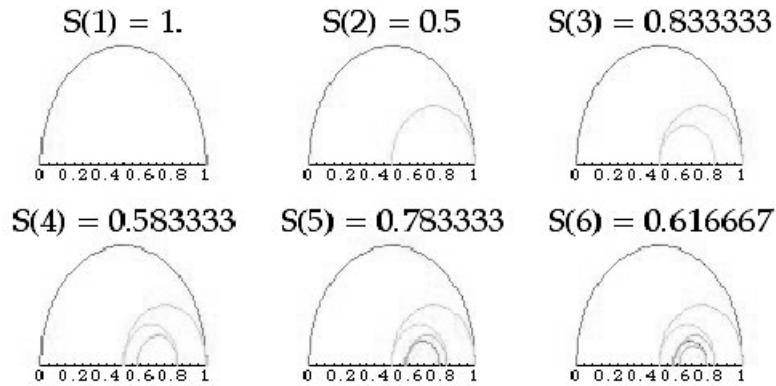
If we take limits as  $n \rightarrow \infty$ , we obtain:

$$\int_1^{\infty} a(x) dx \leq \sum_{k=1}^{\infty} a(k) \leq a_1 + \int_1^{\infty} a(x) dx$$

This is the essence of the integral test, which tells us that the improper integral  $\int_1^{\infty} a(x) dx$  and the infinite series  $\sum_{k=1}^{\infty} a(k) = \sum_{k=1}^{\infty} a_k$  behave similarly. In other words, the infinite series and the improper integral both converge or both diverge. Here, our interest is in the fact that we can use the two improper integrals on the outsides of our inequalities to approximate the sum of the series. Depending upon the size of  $a_1$ , this may give a rather poor window, but finding upper and lower bounds for the sum of the series in this manner for a few problems can aid in the understanding of the nature of approximation.

### Alternating Series

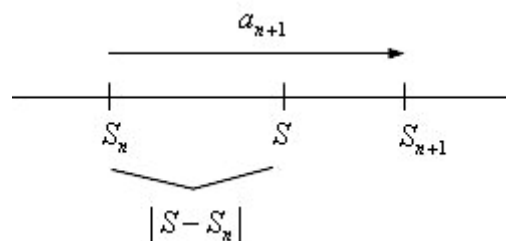
Consider the alternating harmonic series  $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ . Ask students to calculate several terms in the sequence of partial sums  $\{S_n\}$  and then have them plot and label these values on the  $x$ -axis. For most students, plotting these values convinces them that this series will converge. The graphs below offer a visualization of the pattern of convergence of the  $S_n$ .



It is important for students to see that each successive term in the sequence of partial sums is between the two previous terms in this sequence. And consequently,  $S$  must be between any two successive terms as well. For example, we can see that the infinite sum  $S$  above must satisfy  $S_4 < S < S_5$ , and it must also satisfy  $S_6 < S < S_5$ . As  $n$  increases  $a_n = \frac{1}{n} \rightarrow 0$ , and the values of  $S_n$  and  $S_{n+1}$  will become closer together. These values give us both a lower bound and an upper bound for  $S$ , the infinite sum. With  $S$  trapped in between, we are able to find better and better approximations for  $S$ .

It's time to generalize. Consider an alternating series

$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \dots$  where all  $a_k > 0$ , the terms  $a_k$  form a decreasing sequence, and  $\lim_{k \rightarrow \infty} a_k = 0$ . (These conditions are the hypotheses of the Alternating Series Test [AST]. Fortunately, all were true in the previous example.) Assuming these conditions, we can take another look at the relationships between  $a_n$ ,  $S_n$ , and  $S_{n+1}$ . (In the diagram below,  $n$  is an even integer.)



This diagram shows us that the error, the distance between the actual sum  $S$  and the approximation  $S_n$ , must be shorter than the distance represented by  $a_{n+1} > 0$ .

For example, consider  $S = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ . If we approximate  $S$  by

$S_5 = 3019/3600 \approx 0.8386$ , then the error will be less than  $a_6$ , the positive part of the first unused term in the series. That is,  $\text{Error} = |S - S_5| < a_6 = \frac{1}{36} = 0.0278$ . If desired, one can continue to find bounds on  $S$ .

$$\begin{aligned} |S - S_5| &< .0278 \\ |S - .8386| &< .0278 \\ -0.0278 &< S - .8386 < .0278 \\ .8108 &< S < .8665 \end{aligned}$$

This offers a systematic method to find bounds on  $S$ . Alternatively, we can note that  $S$  must also be less than  $S_5$  since  $\frac{1}{36}$  will be subtracted from  $S_5$  to give  $S_6$ . Thus, we may write  $.8018 < S < S_5 = .8386$ . This gives us a smaller interval for  $S$ . One could repeat these calculations for  $S_{10}$  or  $S_{20}$ , for example. Seeing how the error becomes smaller and the window for  $S$  becomes narrower as  $n \rightarrow \infty$  will help to give a feel for error and how quickly the particular alternating series converges. (The AP syllabus asks only for whether a series converges or diverges. However, understanding that some series approach their sum much faster than others can aid in developing a deeper understanding of infinite series.)

Calculating an upper bound for the error of an alternating series that satisfies the conditions of the alternating series test is a rather simple mechanical procedure. Use 12 terms? Find the size of  $a_{13}$ . However, noting and checking the hypotheses of the Alternating Series Test is important, yet often forgotten. To show that a series with both positive and negative terms may converge even if the Alternating Series Test and the error calculation do not apply, it is interesting to use technology to explore the sequence of partial sums for the series  $\sum_{k=1}^{\infty} \frac{\sin(k)}{k}$ . It is important to remind students to check the hypotheses of the Alternating Series Test. In this case,  $\left| \frac{\sin(k)}{k} \right|$  is not a decreasing sequence, nor is  $\frac{\sin(k)}{k}$  even an alternating sequence, although the series does in fact converge.

Another important concept in this study is the relationship between the number of terms and the size of the error. If an alternating series satisfies the conditions of the AST, knowing one effectively determines the other. In the example above, the number of terms was given and an upper bound for the error was found. Alternatively, one could give an upper bound for the error and ask for the number of terms necessary to ensure this. For example, in the series  $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ , if we wish to approximate  $S$  with an error less than  $\frac{1}{200}$ , it is sufficient to use at least  $S_{14}$ , the sum of the first 14 terms, because the size of the 15<sup>th</sup>

term is  $\frac{1}{15^2} = \frac{1}{225} < \frac{1}{200}$ . Note that  $a_{14} = \frac{1}{14^2} > \frac{1}{200}$ , so using  $S_{13}$  may not be adequate to ensure a sufficiently close approximation.

## Taylor Series

The Taylor series for a function  $f$  with infinitely many derivatives at  $x = a$  is given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k = f(a) + f'(a) \cdot (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f'''(a)}{3!} (x - a)^3 + \dots$$

A Maclaurin series is the special case of the Taylor series where  $a = 0$ . Previously, we approximated the sum of an infinite series with the sum of a finite number of terms,  $S_n$ . Here, we approximate a Taylor series with a Taylor polynomial.

To help understand the convergence of Taylor series, add more terms and sketch more graphs! This is essential for seeing how Taylor-series approximations of functions grow better as  $n$  grows larger. What does *better* mean? In this context, it means that for a given value of  $x$  appropriately near  $a$ , the approximation given by evaluating a Taylor polynomial at this  $x$  will grow closer to the actual value of  $f(x)$  as the polynomial is formed using more and more terms of the series. This will be clearer to students when they make calculations for specific values of  $x$ . In the previous discussion of alternating series, we mentioned the relationship between the number of terms and an upper bound for the error. With Taylor series, another piece is included: the value of  $x$ . Knowing two of these three—the value of  $x$ , the number of terms, and an upper bound for the error—will allow us to find restrictions on the third piece of information. Some examples follow.

If the Taylor series is alternating (with the required restrictions) for a given value of  $x$ , then the calculations are the same as for other alternating series. These series offer many opportunities for calculating errors. Common functions with alternating Maclaurin series ( $a = 0$ ) include  $\sin x$ ,  $\cos x$ , and if  $x > 0$ ,  $\ln(1 + x)$ .

For example, let's approximate  $\sin(1)$  with three nonzero terms of the Maclaurin series for  $\sin x$ . We use

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\sin(1) \approx 1 - \frac{1}{3!} + \frac{1}{5!} \approx .842$$

and the error must be less than the absolute value of the fourth term  $\frac{x^7}{7!}$  evaluated at  $x = 1$ ,  $a_4 = \frac{1}{7!} \approx .000198$ . This may be verified by finding  $\sin(1)$  with a calculator.

As another example, assume we wish to use three terms of the Maclaurin expansion for  $f(x) = \ln(1 + x)$  and we want to be sure that the error will be less than 0.005. If we

assume  $x > 0$ , what are the possibilities for  $x$ ? The series for  $f$  is  $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ .

Thus, using three terms means that the upper bound for the error will be  $a_4 = \frac{x^4}{4}$  for

appropriate values of  $x$ . If we force this to be less than 0.005, then this will guarantee that the error will also be less than 0.005. We have  $a_4 = \frac{x^4}{4} < 0.005 \Rightarrow 0 < x < .376$ . (Recall that we assumed  $x > 0$ . If not, then the series won't be alternating, and we cannot use this approach to bounding the size of the error.)

## The Lagrange Error Bound

The situation changes when the series is not alternating. This brings us to the Lagrange error bound. This may appear in either of two forms, depending upon the textbook. Making use of either form effectively comes down to the same idea.

Some texts present *Taylor's Theorem*, a statement that gives the formation of the series and  $R_n$ , where  $R_n$  represents the remainder after the  $n^{\text{th}}$  degree term of the sequence of partial sums of the Taylor series. This is given as  $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$  for some  $c$  between  $a$  and  $x$ , and is often identified as *Lagrange's form of the remainder*. Taylor's Theorem is an existence theorem. The value of  $c$  may never be found, implying that the precise remainder may not be found. We seek instead an upper bound for the remainder, often called the

*Lagrange error bound*. Since we are searching for an upper bound for  $|R_n|$ , where

$|R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!} |x-a|^{n+1}$ , then we need to find an upper bound for  $|f^{(n+1)}(c)|$ , and

then do a few other manageable calculations.

Many other texts begin by asking the reader to consider  $|f^{(n+1)}(x)| \leq K_{n+1}$ , where  $K_{n+1}$  is an upper bound for the  $(n+1)^{\text{st}}$  derivative, often without the restrictions on  $x$  in the previous form. Then they note that the error found by using the  $n^{\text{th}}$  degree Taylor polynomial will be less than or equal to  $\frac{K_{n+1}}{(n+1)!} |x-a|^{n+1}$ . These two forms are essentially the same, even with the varying restrictions on  $x$ , as the following two examples will illustrate.

We can use this approach to find an upper bound for an approximation for  $e^x$ . For example, assume we will use terms of the Maclaurin series through  $n = 8$  to approximate  $e^2$ . Here, with  $a = 0$  and  $x = 2$ , to deal with the ninth derivative, we need to consider the first form with  $0 < c < 2$ . (Without the restriction, there is no upper bound for the derivative of  $f(x) = e^x$ .) We have  $|R_8| = \frac{e^c \cdot 2^9}{9!} < \frac{e^2 \cdot 2^9}{9!}$ . We also know that  $e < 3$  (since we need to assume something!), and  $e^2 < 9$ , so that  $|R_8| = \frac{e^c \cdot 2^9}{9!} < \frac{9 \cdot 2^9}{9!} = .0127$ .

As another type of example, let  $g$  be a differentiable function with  $g^{(n)}(0) = \frac{n}{2n+1}$  and  $|g^{(n)}(x)| \leq n+1$  for all  $x$ . If we calculate the derivatives, we can construct the Maclaurin series for  $g$ , showing terms through  $n = 3$ , to give



$$\begin{aligned}
 &g(0) + g'(0) \cdot x + g''(0) \cdot \frac{x^2}{2!} + g'''(0) \cdot \frac{x^3}{3!} + \dots \\
 &0 + \frac{1}{3} \cdot x + \frac{2}{5} \cdot \frac{x^2}{2} + \frac{3}{7} \cdot \frac{x^3}{3!} + \dots \\
 &\quad \frac{x}{3} + \frac{x^2}{5} + \frac{x^3}{14} + \dots
 \end{aligned}$$

If terms through  $n = 3$  are used to approximate  $g(1/2)$ , we can find an upper bound for the error. This requires finding an upper bound  $K_4$  for  $|g^{(4)}(x)|$ . We have  $|g^{(4)}(x)| \leq 4 + 1 = 5$ . Combining this result with the rest of the Lagrange form gives

$$\text{Error} \leq |g^{(4)}(x)| \cdot \frac{|x|^4}{4!} \leq 5 \cdot \frac{(1/2)^4}{4!} = \frac{5}{384} = .0130$$

Dealing with approximation and error estimates requires patience and a great deal of careful work. Using technology is essential to deepening one's understanding of infinite series. Plotting the sequence of partial sums, graphing a Taylor polynomial along with the original function, or calculating an upper bound for an error—all of these can help to check for reasonableness and, above all, they allow us to more fully see how mathematics works.

## Instructional Unit: Taylor Polynomial Approximation of Functions

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Focus: How to use Taylor polynomials to approximate the value of a function and the error made in that approximation.

Audience: AP Calculus students

### Context

#### Entry Behaviors

Students should be able to compute the derivative and higher-order derivatives of a function at a point. They should also be able to use their graphing calculators to evaluate a function at a point and construct the graph of a function in a specific viewing window. Students should be able to find the maximum value of a continuous function on a given interval.

#### Prior Knowledge

Students should know various functions, their derivatives, and higher-order derivatives. They should also be familiar with what the first and second derivatives say about the shape of the graph of a function. Students should be familiar with factorial function. Students should also understand the Mean Value Theorem.

#### Academic Motivation

It is assumed that students have found approximate values of various functions by using their calculators. The motivation for this topic is to provide an arithmetic technique to approximate the value of a function at a point.

#### Education and Ability

Students should have taken the typical high school mathematics courses of Algebra I, Algebra II, precalculus, and Calculus I or the equivalent of Calculus AB.

#### Performance Setting

#### Social Aspects

This should be used in an AP Calculus classroom where a teacher is guiding the exploration of ideas.

### **Physical Aspects**

The teacher should have the ability to display visual information and written information. Each student should have his or her own graphing calculator.

### **Relevance**

This unit is relevant to anyone who would like to learn how transcendental functions can be approximated arithmetically and with a desire to explore the concepts of Taylor series and power series.

### **Duration**

This unit should take three to five days to complete.

### **Accessibility and Adaptability**

This unit is accessible to anyone with the prior knowledge and a computational device. While graphing calculators are the primary computational tool, anyone with appropriate software could adapt this lesson to his or her setting. Students with a scientific calculator could do most of the work here but might not be able to explore the more geometric parts of the lesson.

### **Goals and Standards**

#### **Essential Question**

There are two essential questions addressed by this unit. How does one approximate numerically the values of a transcendental function, and how much error is made in that approximation?

#### **Goals**

The goal is for students to be able to construct the  $n$ th-degree Taylor polynomial of a function, use it to approximate the value of a function at a point, give upper and lower bounds on that actual value (error in approximation), and decide what degree polynomial is necessary to achieve a certain level of error.

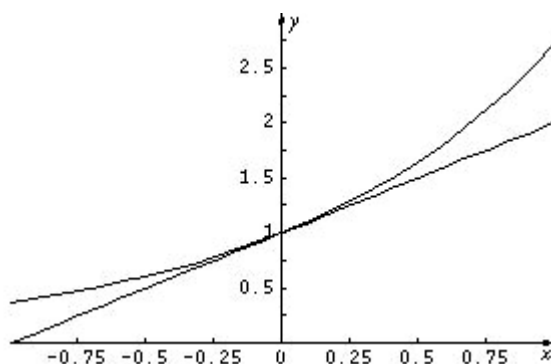
#### **Alignment with the AP Calculus Syllabus**

Students are required by the AP Calculus Syllabus to be able to construct and use Taylor polynomials to approximate the value of a function at a point. They are also required to be familiar with the Lagrange form of the error (remainder) term.

## Taylor Polynomial Approximation of Functions

### Introduction

- Motivation—Approximate the value of  $e^{0.1}$  using only simple arithmetic.  
Use this example to develop the following ideas.
- Recall the approximation of the value of a function using the tangent line (local linearity).  
 $f(x) = f(a) + f'(a)(x - a)$   
Thus, using  $f(x) = e^x$  and  $a = 0$  we could approximate  $e^{0.1} = f(0.1)$  using the tangent line. We get  $f(0.1) \approx f(0) + f'(0)(0.1 - 0) = 1 + 1 \cdot (0.1) = 1.1$ . Students can compare this approximation with a calculator value. Using Maple, I get  $e^{0.1} \approx 1.105170918$ .
- Consider the geometric interpretation of this approximation.



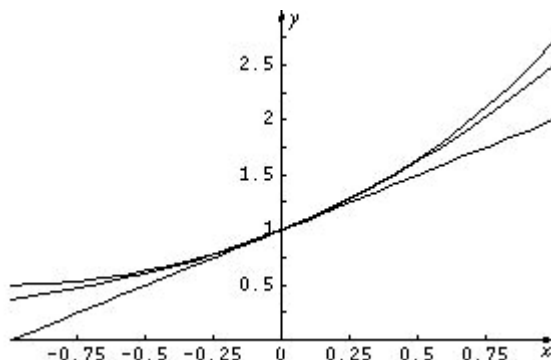
The lower curve is the tangent line given by  $T_1(x) = f(0) + f'(0) \cdot (x - 0) = 1 + x$ . Thus we know that our approximate arithmetical value will actually underestimate the true value of  $e^{0.1}$ .

- We could obtain a better approximation if we could somehow account for the *curve* in the natural exponential function. You could ask students to discuss this curve and indicate how they might determine how much *curve* there is. This should lead to a discussion of concavity and the second derivative.
- The tangent line shares two characteristics of a function: (1) It has the same value as the function at a point, and (2) It has the same slope at that point. In the example above,  $f(x) = e^x$  and  $T_1(x) = 1 + x$  have the same value and same slope (derivative) at  $a = 0$ . What if we now tried to construct a *simple* function (a polynomial) that would have the same value, slope, and concavity (curvature) as  $f(x) = e^x$  at  $a = 0$ ?
- We should try the polynomial  $T_2(x) = a + bx + cx^2$  because we want to satisfy three conditions at  $a = 0$ , and there are three unknowns in  $T_2(x)$ . We want to satisfy
  - $1 = e^0 = f(0) = T_2(0) = a$ ;
  - $1 = e^0 = f'(0) = T_2'(0) = b + 2c \cdot 0 = b$ ; and
  - $1 = e^0 = f''(0) = T_2''(0) = 2c$ .

Thus we have  $T_2(x) = 1 + x + \frac{1}{2}x^2$  and so

$e^{0.1} = f(0.1) \approx T_2(0.1) = 1 + 1 \cdot (0.1) + \frac{1}{2}(0.1)^2 = 1.105$ . Since we already have an approximate value from the calculator, we can compare and see that this is better than the original linear approximation.

7. We can examine this geometrically again.



Students should confirm that the parabolic (second-degree polynomial) approximation is the one that lies between the linear approximation and the actual function.

8. We again have an under approximation since  $T_2''(x) = 1$  and a constant second derivative does not account for the increasing (changing) curvature in the natural exponential function.
9. This should lead to the idea of a third-degree approximation  $T_3(x) = a + bx + cx^2 + dx^3$  satisfying

- i)  $1 = e^0 = f(0) = T_3(0) = a$ ;
- ii)  $1 = e^0 = f'(0) = T_3'(0) = b + 2c \cdot 0 + 3d \cdot 0 = b$ ;
- iii)  $1 = e^0 = f''(0) = T_3''(0) = 2c + 6d \cdot 0 = 2c$ ; and
- iv)  $1 = e^0 = f'''(0) = T_3'''(0) = 6d$ .

This leads, then, to  $T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$  and

$$e^{0.1} = f(0.1) \approx T_3(0.1) = 1 + 1 \cdot (0.1) + \frac{1}{2}(0.1)^2 + \frac{1}{6}(0.1)^3 \approx 1.105166667.$$

10. Ask students to find  $T_4(x)$  and  $T_5(x)$ . Can they generalize to  $T_n(x)$ ?
11. Now use the same approach to approximate  $\sqrt{4.2}$  using the function defined by  $f(x) = \sqrt{x}$ . Note that here we need to adapt to the idea that we are approximating  $f(x) = \sqrt{x}$  near  $a = 4$ . Thus the tangent-line approximation will be  $f(x) \approx T_1(x) = f(4) + f'(4)(x - 4)$ . When we move to the second-degree approximation and others we will use  $T_2(x) = a_0 + a_1(x - 4) + a_2(x - 4)^2$  and  $T_n(x) = a_0 + a_1(x - 4) + a_2(x - 4)^2 + \dots + a_n(x - 4)^n$  in general, respectively. Expanding  $T_n(x)$  in powers of  $(x - 4)$  rather than in powers of  $x$  simplifies the derivation of the coefficients.

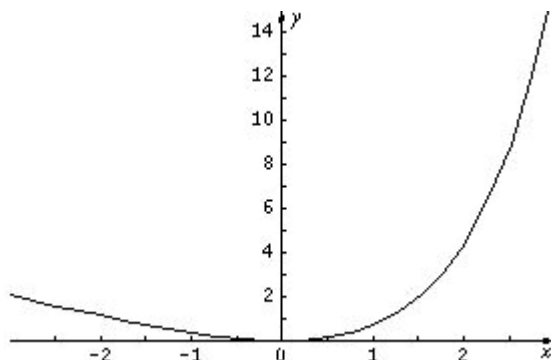
12. This might be a good time for students to find some Taylor polynomials on their own. They could complete the table found in Appendix 1.
13. It's at this point we can verify that the  $n$ th degree Taylor polynomial for a function  $f$  in general is given by

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

To do this, we need to use this formula to show that the value of the function  $f(x)$  agrees with the value of  $T_n(x)$  at  $x = a$  and that the first  $n$  derivatives of these two functions also agree at  $x = a$ . Note that this  $n$ th-degree Taylor polynomial requires  $f$  to be  $n$  times differentiable at  $x = a$ .

### Error Analysis

14. The error made in approximating a function  $f$  by its  $n$ th degree Taylor polynomial  $T_n$  is given by  $R_n(x) = f(x) - T_n(x)$ . One can begin to explore this error using a graphical approach. If we use the example above ( $f(x) = e^x$ ) we would have  $R_2(x) = e^x - (1 + x)$ . Plotting this near  $x = 0$ , we get



Students can use the trace feature on their calculators to find the error at any particular value of  $x$ . We can also see from this graph that if we specify a particular tolerance (maximum error) that this will be obtained for values of  $x$  sufficiently close to. For example, we could construct, or have students construct, the following table for the example above.

Maximum value of error	0.05	0.005	0.0005
Values of $x$ (approximately)	$-0.3338 < x < 0.3004$	$-0.1016 < x < 0.0983$	$-0.0317 < x < 0.0314$

For clarity, this table indicates that  $|R_1(x)| < 0.005$  when  $-0.1016 < x < 0.0983$ .

15. We can continue to explore this error geometrically and numerically for higher-degree Taylor polynomials. For the example  $f(x) = e^x$  with  $a = 0$ , we can construct the table

Degree of polynomial $n$	Maximum error for $-1 \leq x \leq 1$	Interval with error $< 0.005$	Graph of $R_n(x)$
2	0.21828183	$-0.31889 < x < 0.30280$	
3	0.05161517	$-0.60595 < x < 0.57134$	
4	0.00994850	$-0.92978 < x < 0.87551$	

We can have students construct such tables for various functions at particular values of  $a$ . A worksheet for them to complete is Appendix 2.

16. It would also be appropriate to show students a few applets that are available for Taylor polynomial approximation. Some sites to consider:
- <http://www.math.jhu.edu/~jrm/vander/stable/TPTest.html>
  - <http://mathforum.org/mathtools/cell/c,15.18.1,ALL,ALL/>
  - <http://www.hostsrv.com/webmaa/app1/MSPScripts/webm1010/taylor.jsp>
  - <http://www.ma.utexas.edu/cgi-pub/kawasaki/plain/infSeries/6.html>

### Further Analysis of Error

17. If we continue to work with the initial example and examine the graphs of the errors for the various degrees of the Taylor polynomials, it appears that the graph of the error for  $n = 1$  is quadratic in nature, the graph of the error for  $n = 2$  is cubic in nature, the

graph of the error for  $n = 3$  is quartic in nature, etc. To verify this you can do *power zooming* on the calculator as explained in the earlier article “The Tangent Line as ‘The Best’ Linear Approximation.” Thus students can have some idea of the nature of the error that is incurred with a Taylor polynomial approximation of a function prior to seeing the Lagrange form for the remainder. In general, one should achieve an “ $n + 1$ ” degree error when using  $T_n$ . This won’t always be true; for example, a third-degree approximation for the *sin* function at  $a = 0$  will give an error of fifth-degree nature rather than fourth-degree nature.

18. Further analysis of the error can be done symbolically through the Lagrange form for the remainder. This says that the error made in approximating  $f(x)$  by the  $n$ th degree Taylor polynomial  $T_n(x)$  is given by  $R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-a)^{n+1}$  for some  $c_x$  between  $a$  and  $x$ . Proof of this is given in Appendix 3, along with a corollary that allows us to determine what degree Taylor polynomial is needed to guarantee a certain error in approximation on a given interval. Students are required to know the Lagrange form for the error made in approximation but are not responsible for the proof of this.
19. Let’s consider the original example and use the corollary to the theorem in Appendix 3. Suppose we wish to approximate  $f(x) = e^x$  by a Taylor polynomial centered at  $a = 0$  with a maximum error of 0.00005 on the interval  $[-2, 2]$ . We know that  $f^{(n+1)}(x) = e^x$ , so on the interval  $[-2, 2]$  we will have  $e^{-2} \leq f^{(n+1)}(x) \leq e^2$  giving  $|f^{(n+1)}(x)| \leq e^2$  on  $[-2, 2]$ . The maximum error of 0.00005 will be assured if we have  $\frac{e^2}{(n+1)!} \cdot 2^{n+1} < 0.00005$ . By just checking values of  $n$  in this formula, this analysis leads to a minimal degree polynomial of degree  $n = 11$ . Thus  $T_{11}(x)$  will approximate  $f(x) = e^x$  with an error less than 0.00005 for all  $x$  in  $[-2, 2]$ . Appendix 4 gives a worksheet that students can use to do this kind of analysis. The table is incomplete so that they can choose their own values.
20. Note that the Lagrange error formula can also be used to estimate the maximum error achieved by using the  $n$ th-degree Taylor polynomial in approximation for a particular value of  $n$  in place of the graphical approach to finding this error. However, this method of analysis might lead to an estimated error that is much greater than the actual error obtained.



## Appendix 1: Taylor polynomials Worksheet

Find the specified Taylor polynomials at the given value of  $a$ .

Function	$a$	$T_1(x)$	$T_2(x)$	$T_3(x)$	$T_4(x)$	$T_n(x)$
$f(x) = \frac{1}{x}$	1					
$f(x) = \sin(x)$	0					
$f(x) = \cos(x)$	0					
$f(x) = \ln(1 + x)$	0					
$f(x) = \cos(x)$	$-\pi$					
$f(x) = \frac{1}{(1-x)}$	0					
$f(x) = \ln(-x)$	-1					
$f(x) = 1 + x + x^2 + x^3$	4					

## Appendix 2: Error in Taylor Polynomial Approximation Worksheet

$f(x) =$			$a =$
Degree of polynomial $n$	Maximum error for $-1 \leq x \leq 1$	Interval with error $< 0.005$	Graph of $R_n(x)$

## Appendix 3: Proof of Taylor's Theorem with Lagrange's Remainder

### Taylor's Theorem with Lagrange's Remainder

Let  $f$  be  $n + 1$  times differentiable on the interval  $(a - r, a + r)$ . For each  $x$  in  $(a - r, a + r)$  there is a point  $c_x$  between  $a$  and  $x$  so that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \frac{f^{(n+1)}(c_x)}{(n + 1)!}(x - a)^{n+1}.$$

**Proof:**

Fix a value of  $x \neq a$  in  $(a - r, a + r)$ , and choose  $K$  so that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \frac{K}{(n + 1)!}(x - a)^{n+1}.$$

Define a function  $g$  on the closed interval from  $a$  to  $x$  by

$$g(t) = f(x) - f(t) - \frac{f'(t)}{1!}(x - t) - \frac{f''(t)}{2!}(x - t)^2 - \frac{f'''(t)}{3!}(x - t)^3 - \cdots - \frac{f^{(n)}(t)}{n!}(x - t)^n - \frac{K}{(n + 1)!}(x - t)^{n+1}.$$

Now  $g(x) = 0 = g(a)$  and since  $g$  is differentiable on the open interval from  $a$  to  $x$ , by Rolle's Theorem there is some point  $c_x$  between  $a$  and  $x$  with  $g'(c_x) = 0$ . We see that

$$g'(t) = f'(t) - (-f'(t)) - f''(t)(x - t) - \cdots - \left( -\frac{f^{(n)}(t)}{(n - 1)!}(x - t)^{n-1} \right) - \frac{f^{(n+1)}(t)}{n!}(x - t)^n - \left( -\frac{K}{n!}(x - t)^n \right).$$

Note that this is a collapsing sum and simplifies to  $g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x - t)^n + \frac{K}{n!}(x - t)^n$ . With  $g'(c_x) = 0$  we get  $K = f^{(n+1)}(c_x)$ , the desired result.

### Corollary

Let  $f$  be  $n + 1$  times differentiable on the interval  $[a - r, a + r]$  with  $|f^{(n+1)}(x)| \leq M$  for all  $x$  in  $[a - r, a + r]$ . Then  $|R_n(x)| = |f(x) - T_n(x)| \leq \frac{M}{(n + 1)!} |x - a|^{n+1}$  for all  $x$  in  $[a - r, a + r]$ .

**Proof:**

This comes immediately from the theorem above by replacing  $f^{(n+1)}(c_x)$  with  $M$ .

**Special Focus: Approximation**

Note: The proof of the theorem above is given in *Calculus with Analytic Geometry*, 5th Edition, by C.H. Edwards and David E. Penney, Prentice Hall Inc., 1997. As noted there, the proof uses a “trick” suggested by C. R. MacCluer.

## Appendix 4: Finding the Degree for Desired Error Worksheet

Function	$a$	Interval	Maximum Error Desired	$f^{(n)}(x)$	$n$ guaranteed to achieve error
$f(x) = \sin(x)$	$a = 0$	$[-\pi, \pi]$	0.00005		
$f(x) = \ln(x)$	$a = 1$	$[\frac{1}{2}, 2]$	0.0005		
$f(x) = \sqrt{x}$	$a = 4$	$[1, 9]$	0.0001		

## AP Calculus Special Focus Material: Approximation

### Chief Editor:

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Steve is a professor of mathematics at Bloomsburg University. His research interests include the statistical analysis of cancer chemoprevention experiments and computer forensics. Steve has published papers in *Biometrics*, *Anticancer Research*, and *Computer Methods and Programs in Biomedicine*; presented his results at national conferences; written several books; and has been awarded grants from the National Science Foundation, the Center for Rural Pennsylvania, and the Ben Franklin program. He is a nationally endorsed consultant for the College Board and Exam Leader for the grading of the AP Calculus Exam, and he usually teaches courses in calculus and statistics. Steve regularly works with Mathematica and TeX, and is busy writing an introductory statistics text to be published by W.H. Freeman.

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Mark Howell teaches AP Calculus at his alma mater, Gonzaga High School, a Jesuit boys' school in Washington, D.C. Over his 28-year career there, he has taught math classes from algebra through calculus, as well as AP Computer Science. He has authored many articles and activities involving the use of handheld technology in the classroom, and has spoken about it at conferences all around the world. He served at the AP Calculus Reading for 12 years, as a Reader, Table Leader, and Question Leader. He served on the AP Calculus Development Committee from 1997 to 2000. More recently, Mark has “appeared” at College Board online events, including introductory and intensive workshops in AP Calculus. Mark is the coauthor of the recently published title *Be Prepared for the AP Calculus Exam* (Skylight Publishing), as well as the most recent *Teacher's Guide for AP Calculus*.

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Monique Morton has been teaching mathematics at Woodrow Wilson Senior High School, a District of Columbia public school, since 1987. A recipient of the 2001 Siemens Award for Advanced Placement and a state-level finalist for the Presidential Award for Excellence in Mathematics and Science Teaching, she has been a Reader, a Table Leader, and a Question Leader for AP Calculus. A member of the AP Calculus Development Committee, Ms. Morton is also a member of the SAT Mathematics Test Development Committee, a National Workshop Leader for Mathematics Vertical Teams, and a College Board-certified consultant for AP Calculus and Pre-AP Mathematics.

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John Bapst Memorial High School  
Bangor, Maine

Brendan Murphy has taught for the last 16 years at the college, high school and middle school levels. He is a National Leader for the College Board and is an endorsed consultant for AP Calculus, AP Statistics, and Pre-AP Mathematics. In 2005, Brendan was a finalist for the Maine Teacher of the Year, and was a 2005 Siemens Award Winner for Advanced Placement. In 2006 he was one of two finalists for the Presidential Award for Excellence in Mathematics in Maine.

Brendan has run weeklong summer institutes in AP Calculus, AP Statistics, and Vertical Teams Math. He has presented numerous workshops for the College Board around the country and has been a speaker at many state and national conferences. For the past two



years Brendan has run a mentoring program for new AP Statistics teachers across Maine for the Maine Department of Education. He is a leader in distance learning and was featured in *Newsweek* magazine in May 2005.

Brendan has a BSCE from Tufts University as well as an MBA from Bentley College. He has a summer house in Bingham, Maine, and in his spare time is a licensed whitewater rafting guide. He enjoys traveling and has rafted on 75 rivers in five countries. Brendan is currently teaching at John Bapst Memorial High School in Bangor, Maine, and is active in the Maine Association of Math Teams.

Larry Riddle  
Agnes Scott College  
Decatur, Georgia

Larry Riddle is a professor of mathematics at Agnes Scott College, a liberal arts college for women located in metropolitan Atlanta. He served as the Chief Reader for the AP Calculus Exam from 2000 to 2003, and became chair of the AP Calculus Development Committee in 2005. His interest in the use of technology in teaching mathematics has led him to develop a software program called *IFS Construction Kit* for teaching about the fractals associated with iterated function systems. He and his students have also constructed an extensive Web site containing biographies of more than 170 woman mathematicians that demonstrate the numerous achievements of women in mathematics.



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