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# The Teaching Series

## Special Focus in Calculus

Differential Equations

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**AP Central is the official online home for the AP Program and Pre-AP: apcentral.collegeboard.com.**

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***Important Note:***

*The following materials are organized around a particular theme that reflects important topics in AP Calculus. They are intended to provide teachers with professional development ideas and resources relating to that theme. However, the chosen theme cannot, and should not, be taken as any indication that a particular topic will appear on the AP Exam.*

# Differential Equations in Advanced Placement Calculus

Thomas Dick
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Starting in 1995, the Advanced Placement Program in calculus began requiring the use of graphing calculators on some parts of the examinations. Shortly after, the Course Descriptions for Advanced Placement Calculus underwent their greatest changes in 30 years (when distinct Calculus AB and Calculus BC Course Descriptions were first described). These changes, effective with the 1997-98 academic year, were intended to respond to a variety of evolutionary forces. The previous decade had seen unprecedented activity in new calculus curriculum development, and the new AP Course Descriptions for Calculus AB and BC reflect some of the most important trends that emerged from that activity.

Such changes in both the technology requirements for the examinations and the Course Descriptions themselves put extraordinary demands on AP Calculus teachers to learn new tools, techniques, and topics. It is simply remarkable how well the AP Calculus teacher community has responded to this challenge. While there has been some fine-tuning of the graphing calculator requirements for the examinations from year to year, there have been no substantive changes in the Course Descriptions since 1998 until recently. Effective with the 2003-04 academic year, the Calculus AB Topic Outline will include slope fields, a topic that was first introduced in the Calculus BC Topic Outline with the major changes of 1997-98.

While the addition of this single topic might seem to be a fairly minor change, especially in comparison to the dramatic changes made six years previously, it is worth noting that the vast majority of AP teachers had never encountered slope fields before its appearance in the 1998 Calculus BC Course Description. Slope fields (or direction fields) provide a visualization technique associated with the study of differential equations, and they can serve as a powerful conceptual tool for students and teachers alike to see new connections. Indeed, I will never forget how one teacher described his own introduction to slope fields as an “epiphany” in terms of the new understandings and insights it enabled.

Slope fields can be thought of as wedding two fundamental themes in AP Calculus. One of these themes is that of *local linearity*—differentiable functions have graphs that locally are approximately “straight.” This simple idea can be appreciated directly and visually through zooming in sufficiently on the graph of a differentiable function (an activity made dynamic and accessible with graphing calculator technology). While simple, the idea is nevertheless profound. Indeed, the mathematician and mathematics educator David Tall goes so far as to describe local straightness as a *cognitive root* for differentiability in the sense that it powerfully suggests so many of the important mathematical consequences of differentiability. For example, a remarkably straightforward motivation for L'Hôpital's Rule can be presented by appealing to the idea of local linearity.

The second theme is that of *multiple representations*. Some know this as the "rule of four," meaning that functions have representations as tables, graphs, formulas, and words. To be a competent user of calculus means more than manipulating symbols. It means being able to effectively work with tables of numerical data, with graphical plots, with verbal descriptions of physical situations, and with symbolic formulas and equations. Moreover, it means making and using conceptual connections between these various representations. In this regard, slope fields provide a graphical visualization tool for differential equations based on the principle of local linearity. A slope field is essentially an organized collection of several close-up snapshots of a differentiable function's graph (i.e., line segments of various slopes) obtained by simply extracting the slope information that can be gleaned from a differential equation. Like the picture that emerges from the many individual tiles in a mosaic, a picture of the graphical behavior of solutions to a differential equation can emerge from the display of these line segments over a sampling of points in the plane, usually a rectangular lattice of equally spaced points. (In the same way, Euler's Method—still a Calculus BC topic, but not a Calculus AB topic—can be viewed as exploiting the idea of local linearity to produce a numerical table of approximations to the solution of a differential equation with an initial condition.)

More generally, the addition of slope fields to the Calculus AB Topic Outline should be seen as underlining the importance of differential equations as both a unifying theme in calculus as well as an introduction to a vibrant "cutting edge" of calculus applications. Separable differential equations have been an important topic in both Calculus AB and BC for many years, long before the major changes of 1997-98. The standard symbolic technique for solving such equations remains an important skill, and problems requiring it are generally tested in the "calculator-closed" part of the AP Examinations. More emphasis is being placed on modeling physical situations with differential equations, and in Calculus BC special attention is given to the logistic differential equation as an important special application.

The language of differential equations can and should be introduced very early in calculus, as differential equations appear and re-appear naturally throughout the course. For example, implicit differentiation results in relations that are differential equations, related rates problems involve differential equations, and of course, techniques of antidifferentiation are essentially special cases of solving differential equations. The most important function models arise as solutions to very simple differential equations. For example, a function whose rate of change is directly proportional to function value, $dy/dt = ky$, is the differential equation underlying exponential growth and decay.

Slope fields can be used very effectively in taking a graphical approach to the idea of an antiderivative, and therein lies the primary justification for incorporating them into the Calculus AB Course Description. In turn, the biggest "punchlines" of calculus, the Fundamental Theorems, can be appreciated in new ways using slope fields.

## Differential Equations: Multiple Representations, Solutions, and Teaching Opportunities

The materials included here are built around the theme of differential equations and provide insight and suggestions for how to support student understanding and appreciation of this important topic. We chose differential equations as the theme for the year because of the power of differential equations and the ability to look at them throughout the Calculus AB and Calculus BC courses. Students often have difficulty with questions involving differential equations on the AP Exam. Hopefully, these materials will support both teachers and their students as they work together.

A team of experienced AP Calculus consultants wrote these materials. The team includes a former chair of the AP Calculus Development Committee and others who are past or present members of the Development Committee, a former Chief Reader for AP Calculus, as well as those who were Exam Leaders, Question Leaders, and Table Leaders at the Reading. I am very grateful for their significant contributions.

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We also wish to thank Jim Hartman, Ben Klein, and Nancy Stephenson for their ideas.

AP Central includes other resources to support the differential equations theme. These can be found on the Calculus AB and Calculus BC Course Home Pages:

apcentral.collegeboard.com/calculusab
apcentral.collegeboard.com/calculusbc

- On the Exam Questions pages, you will find the free-response questions, scoring guidelines, sample student responses, and scoring commentary for recent years. Several of these exams include questions involving differential equations (some with slope fields and Euler's Method).

- The AP Calculus Course Home Pages have many interesting articles, and among them is an article and classroom-ready handout about slope fields written by Nancy Stephenson, a recent member of the AP Calculus Development Committee.
- You can search the Teachers' Resources area for reviews of teaching resources for differential equations, slope fields, and Euler's Method. Included are textbooks, books, software, calculators, videos, CD-ROMs, Web sites, etc. A sample of some of those reviews is included at the end of this section.
- A recent AP Central online event, "The Graphical Approach to Differential Equations," presented by Lin McMullin, is archived on AP Central and provides valuable teaching tips for differential equations and slope fields.

New content is added to AP Central regularly, so visit often!

***Important Note: Within these materials, references to particular brands of calculators reflect the individual preferences of the respective authors; mention should not be interpreted as the College Board's endorsement or recommendation of a brand.***

My best wishes for successful and rewarding experiences teaching AP Calculus.

Susan Kornstein
Director, Content Development
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## Introduction

There are, generally speaking, two different types of problems in an AP Calculus course. In one type, you may be given a function and then asked about its rate of change; in the other type, you are given how the function changes and then asked to identify the function. Thus the notions of derivative and antiderivative permeate the course. The term *differential equations* may seem formidable at first, referring to a section far into the textbook that covers special techniques of finding antiderivatives, usually in an applied setting. But since a differential equation is nothing more than an equation that involves a derivative, differential equations occur throughout the course. A solution to a differential equation is simply a function that satisfies the equation. These materials will present some ideas for dealing with differential equations throughout the course so that students feel more comfortable with the topic than they will if it is hurriedly covered at the end of the course. The materials will also show how the new AB topic of slope fields can serve as a unifying bridge between implicit differentiation and differential equations.

**Early Differential Equations**

First, let's explore how you can introduce the terminology of differential equations and what is meant by a solution to a differential equation early on in the course. Usually, one thinks of being handed a differential equation with the task of solving it. But remembering what a differential equation is, we see that when students are first learning the rules of differentiation, they are producing differential equations. This presents an opportunity to give students drill work practicing these rules, and as a bonus, at the same time we can give them practice dealing with parameters. Here are some of the types of questions we are proposing.

Solutions to all numbered examples are provided on page 21.

**Example 1 (only requires knowledge of the power rule):**

Find all values of $p$ so that $y = x^p$ is a solution to the differential equation $3x\, dy/dx = y$.

**Example 2 (involves use of chain rule):**

Find all values of $B$ so that:

$y = \sqrt{Bx+3}$ is a solution to $\dfrac{dy}{dx} = \dfrac{5}{y}$ .

**Example 3 (uses exponential function, chain rule, and higher order derivatives):**

Find all values of $A$ so that $y = 7e^{Ax}$ is a solution to $d^2y/dx^2 = 9y$.

You can see that questions like this are easy to generate and allow students an easy entry into the meaning of a solution to a differential equation.

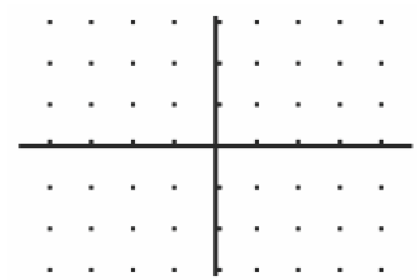
**Implicit Differentiation and Slope Fields**

The familiar topic of implicit differentiation offers an early opportunity to introduce the idea of a slope field. Consider the problem of finding the slope of the line tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4). Using implicit differentiation, we find $dy/dx = -x/y$ and so the slope is $-3/4$. Note however that the 25 has nothing to do with $dy/dx$.We would have obtained $dy/dx = -x/y$ no matter what circle we had started with. Thus $dy/dx = -x/y$ gives a number at each point of the plane where $y \neq 0$; this number is the slope at that point of the circle, centered at the origin, through the point. If we pick a lattice, or grid, of points $(x, y)$ and draw a short line segment through each point with slope $-x/y$, we have constructed what is known as a slope field. Keeping in mind that circles centered at the origin have vertical tangents at points on the $x$-axis (that is, where $y = 0$), we draw short vertical lines when $y = 0$.

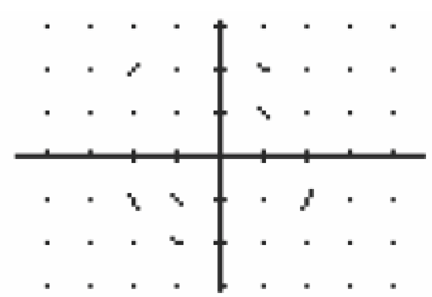
**Classroom Activity**

You can construct a slope field for $dy/dx = -x/y$ as a classroom activity in the following way.

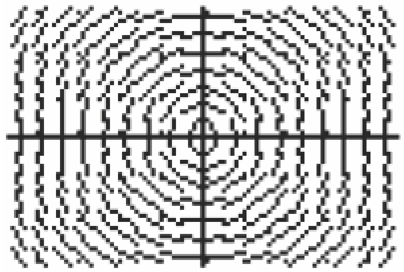
First, make a grid by clearing the *y equals* screen and putting the grid on. Graph in a friendly window so that the screen looks like the screen below.



Project the screen onto the chalkboard. You can assign a particular point to each individual in your class. If you have just a few students, you may want to give each student two or three points. The student calculates the slope of the curve at her point and then comes to the board and draws a small line segment with the desired slope at her point. If the student's point is (2, –1), then the student locates the point (2, –1) and at that point draws a small line segment of slope 2. Soon the slope field will look like the picture below.



When enough "slope segments" have been drawn, your students should recognize that the picture looks like "a bunch of circles." Here is a place that the technology of the graphing calculator lets us construct many similar pictures without this time-consuming effort. We do not have to draw slope fields like this for every problem. The TI-83 Plus has a slope field "app" that may be obtained online, and a program for the TI-83 is included on page 30. Both the TI-89 and the TI-86 have a built-in differential equation mode that draws the slope field when the differential equation is entered in the *y equals* screen. Using the calculator, we get a screen that looks like:



Of course, we started with knowing that the curves satisfied $x^2 + y^2 = r^2$ and so were circles centered at the origin, but now we see these curves again as our eyes "fill in" the patterns.

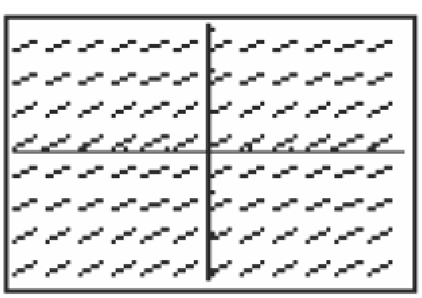
We can also use this moment to talk about higher order derivatives with implicit differentiation. We don't want to get too complicated, but starting with $dy/dx = -x/y$, we saw that the slope at the point (3, 4) is negative, and so the solution function through (3, 4) is decreasing. What can we say about its concavity? Starting with $dy/dx = -x/y$, we differentiate both sides with respect to $x$. We get:

$$\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{-1 \cdot y - (-x)\left(\frac{dy}{dx}\right)}{y^2}$$

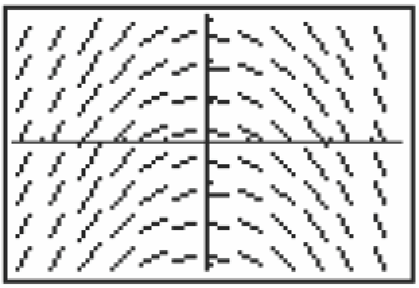
Simplifying, we have $d^2y/dx^2 = \dfrac{-y + x\left(\dfrac{-x}{y}\right)}{y^2} = \dfrac{-(y^2 + x^2)}{y^3}$.

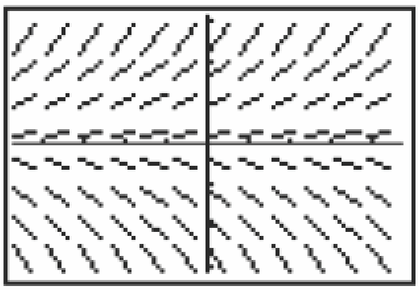
Thus at the point (3, 4), the second derivative is negative, and the curve is concave down. Indeed, we can see that the sign of the second derivative is the opposite of the sign of $y$. This is also what we "see" when we look at the slope field.

We'll look at some more examples that we can discuss even before the concept of integration has been introduced. We must use some equations that are easy to handle. Here are a few such slope fields.



This is generated by $dy/dx = 1$.

 The equation here is $dy/dx = -x$.

 Here we have $dy/dx = y$.

Students will know that the functions that solve these differential equations are $y = x + C$, $y = -x^2/2 + C$, and $y = Ce^x$ respectively. They may have a little bit of trouble with the constants, but it's manageable even at this early stage. They can then match their knowledge of the solutions to the picture of the solutions they get by "filling in" the gaps in the slope field. Again, you can ask students to compute the second derivatives and interpret their answers in terms of concavity.

So far we have talked about what a slope field is, how to draw one, and about the connection to implicit differentiation. Certainly on the free-response portion of the AP Examination students may be asked to draw a few lines of a slope field on a grid, as we did when first starting the topic. Other questions may ask students to sketch a solution curve when they are given a differential equation (such as 2002 BC 5) or draw a conclusion from a slope field they were given (such as 2000 BC 6). How might multiple-choice questions be framed? Some possible ways include the following four constructions:

- Given a differential equation, choose the correct slope field.
- Given a solution function, choose the correct slope field.
- Given a slope field, choose the correct differential equation.
- Given a slope field, choose the correct solution function.

Other types of slope field questions may be included on future exams.

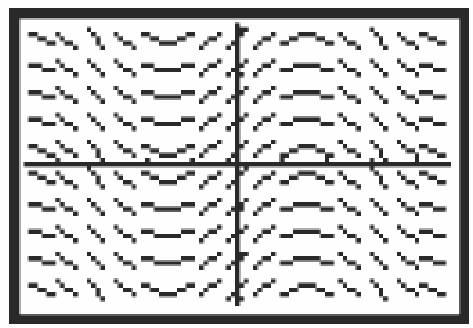
What are features of a slope field that students should look for?

**Reading a Slope Field**

A few features are easy to identify and help sort out most problems:

- Look for the places where the slopes are 0; that is, $dy/dx = 0$.
- Look at the slopes along the $x$-axis.
- Look at the slopes along the $y$-axis.
- Look to see if the slopes only depend on $x$.
- Look to see if the slopes only depend on $y$.
- Look to see where the slopes are positive and where they are negative.

Let's put these ideas into practice.



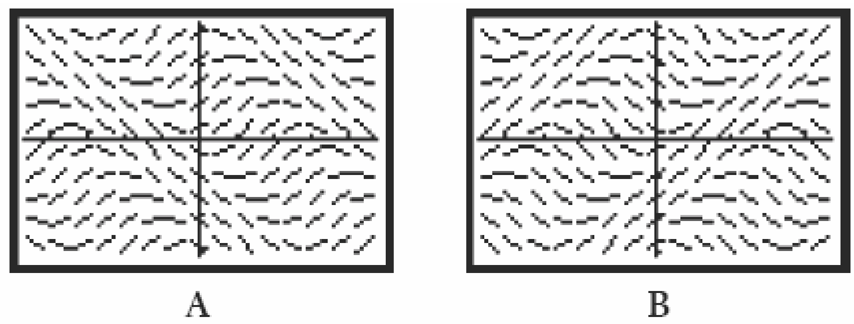
The slope field for the differential equation $dy/dx = \cos(x)$ is shown at the right. As expected, the slopes suggest parallel graphs of functions of the form $y = \sin(x) + C$, giving us a nice visualization of what we know to be the general solution. As useful as this visualization might be, though, slope fields are far more useful than that, as we hope to show in this little exploration.

A slope field gives us useful information about the solution to a differential equation even when we are unable to "solve" the differential equation itself. By analyzing the *slopes* themselves as they relate to the differential equation, we get an enhanced understanding of how much information a differential equation can convey.

Try this exploration to see how well you can match a differential equation to its slope field purely on the basis of the differential equation itself. Caution: do not attempt to solve the differential equations; they are much harder to *solve* than they look.

**Example 4:**

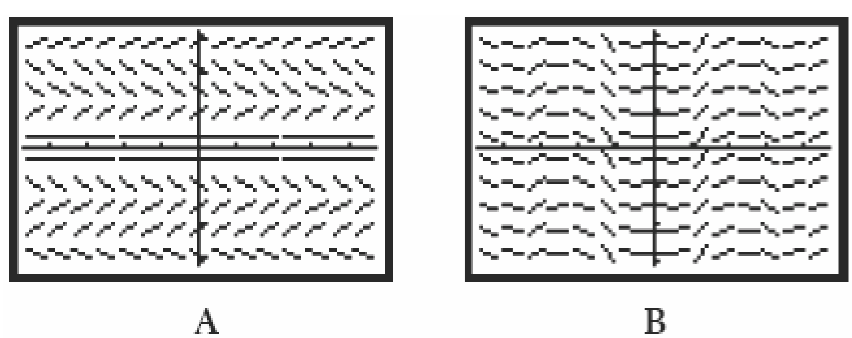
One of the following slope fields is for the differential equation $dy/dx = \sin(x + y)$, and the other is for the differential equation $dy/dx = \sin(x - y)$.



1. Where would you expect the slope to be zero in the slope field for $dy/dx = \sin(x + y)$? Where would you expect the slope to be zero in the slope field for $dy/dx = \sin(x - y)$?
2. Which slope field goes with which equation?
3. One of the graphs shows "stripes" with slope 1, the other shows stripes with slope −1. Why?
4. Can you determine where $\pi$ and $-\pi$ would be on the $x$-axis in both graphs?

**Example 5:**

One of the following slope fields is for the differential equation $dy/dx = \sin(x^3)$, and the other is for the differential equation $dy/dx = \sin(y^3)$.



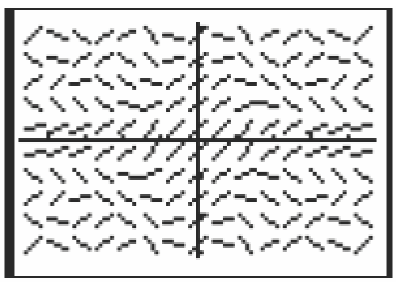
1. Which slope field shows slopes that depend on $x$ but not on $y$? Which slope field shows slopes that depend on $y$ but not on $x$?

2. Which slope field goes with which equation?

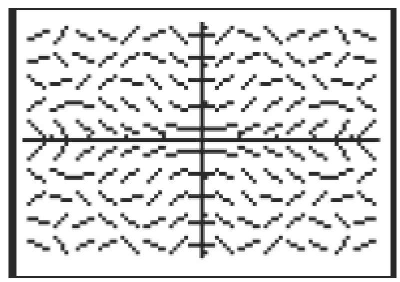
3. The slope field for $dy/dx = \sin(x^3)$ is symmetric with respect to one of the axes. Which axis, and why? What kind of symmetry would be expected of the slope field for $dy/dx = \sin(y^3)$?

**Example 6:**

One of the following slope fields is for the differential equation $dy/dx = \cos(xy)$, and the other is for the differential equation $dy/dx = \sin(xy)$.



A



B

1. Which slope field goes with which equation? How can you tell?

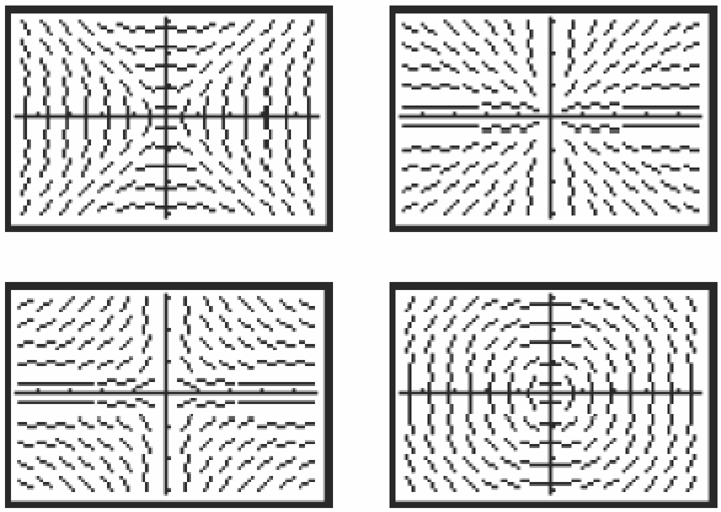
2. Explain how the behavior of the slope lines along the axes helps to identify the correct equation.

3. Explain how the symmetry of the graph helps to identify the correct equation.

**Example 7:**

Match the slope fields below to one of the following differential equations:

$$\frac{dy}{dx}=\frac{y}{x},\ \frac{dy}{dx}=\frac{x}{y},\ \frac{dy}{dx}=-\frac{y}{x},\ \text{or}\ \frac{dy}{dx}=-\frac{x}{y}\ .$$



A warning and a word of sympathy: it is easy to choose windows that give little information about the slope field. Do not become discouraged as you start to use your calculator to explore these ideas.

Some of these examples cover slope fields for differential equations for which we have no formula for the solutions, yet we are able to get some idea of the behavior of a solution by looking at what appear to be natural "flow lines." When covering implicit differentiation, it is not easy to see what some of the curves look like. We might think about using the slope field corresponding to *dy*/*dx* in order to get an idea of the behavior of the curve we started with, but could not draw. The picture can be reinforced by analyzing the signs of *dy*/*dx* and $d^2y/dx^2$.

We now look at what we can do with differential equations later in the course, after antidifferentiation has been covered.

**Separable Differential Equations**

We return to the simple example of $dy/dx = -x/y$ that we started with. Following the method given in textbooks, we separate the variables and integrate:

$\int y\,dy = \int -x\,dx \Rightarrow\ y^2/2 = -x^2/2 + C_1.$

We can rewrite this last equation as $x^2 + y^2 = C$, but now we must remind ourselves of what we are trying to do. We are trying to find a function that satisfies the differential equation, and circles are not the graphs of functions. Thus we need to think of the solutions as functions of the form

$y = \sqrt{C - x^2}$ or $y = -\sqrt{C - x^2}$ .

If we are given a particular point the solution must go through, then we must solve for $C$ and choose one of the square roots (see Example 8).

Let's just say a word about the mystery of integrating one side with respect to $y$ and the other side with respect to $x$. You're not supposed to do different things to different sides of an equation! Actually, what we have done is multiply through by $y$ and then integrate both sides with respect to $x$. This is what really happened:

$y\,dy/dx = x \Rightarrow \int y\,dy/dx\,dx = \int x\,dx$ .

Now on the left side we are looking at the chain rule, and so we use the substitution $y = y(x)$ to arrive at the form $\int y\,dy$ (see Example 8).

Students should be able to handle separable differential equations in a variety of ways.

**First**: Students should be able to separate variables. We look at a generic example.

**Example 8:**

Find the solution to the differential equation $dy/dx = x^2 \sin(x^3)/2y$ that passes through the point (2, 7). This example is a typical "twofer," in which the students have to do two antidifferentiations in one problem.

**Second**: Students should be able, *after having seen the solution carefully established in class*, to recognize certain equations. Specifically, students can recognize that the general solution to $dy/dx = k \cdot y$ is $y = A_0 \cdot e^{kx}$. A variant of this equation comes from Newton's Law of Cooling and similar equations that come from other examples, like mixing problems. (See page 48 for a Computer Based Laboratory experiment exploring Newton's Law.) This differential equation takes the form $dy/dx = A \cdot y + B$, where $A$ and $B$ are constants. Thus the right side is simply a linear function of $y$. Recognizing this allows us to see that the solution is a linear function of the exponential function. We go through this with some simple numbers.

**Example 9:**

Find the particular solution to $y' = 3y + 6$, with $y(0) = 7$.

Students who can recognize what the solution must be will be at an advantage even if the directions force them to obtain the solution by separating variables.

**Third**: Students should be able to use the Fundamental Theorem of Calculus as an accumulator of change.

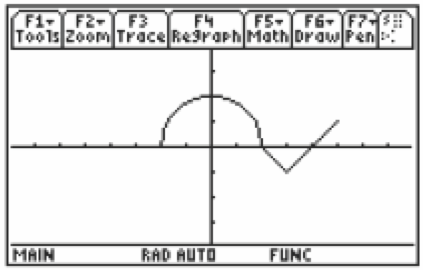
**Example 10:**

If $y' = \sin(x^2)$, and $y(3) = 7$, find $y(5)$.

This approach can also be used in conjunction with the calculator to answer questions of the form: "Here is the derivative. What does the function look like?"

**Example: 1997 AB/BC 5**

The following graph was provided, and the question is repeated below the graph.




The graph of a function *f* consists of a semicircle and two line segments as shown above.

Let *g* be the function given by $g(x) =$

$$\int_0^x f(t)\,dt \,.$$

a) Find $g(3)$.
b) Find all values of $x$ on the open interval (-2, 5) at which *g* has a relative maximum. Justify your answer.
c) Write an equation for the line tangent to the graph of *g* at $x = 3$.
d) Find the $x$-coordinate of each point of inflection of the graph of *g* on the open interval (-2, 5). Justify your answer.

**Solution**

a) Using geometry and the area of a quarter circle and a triangle, and remembering that the triangle is below the $x$-axis, we have $g(3) = \pi - 1/2$.

b) *g* has a relative maximum at $x = 2$ because $g'$, which is *f*, changes sign from positive to negative at $x = 2$.

c) $g(3) = \pi - 1/2$ and $g'(3) = f(3) = -1$. Therefore the tangent to *g* at $x = 3$ is $y - (\pi - 1/2) = -(x - 3)$.

d) *g* has points of inflection at $x = 0$ and $x = 3$ because the slope of $g' = f$ changes sign at $x = 0$ and $x = 3$. Put another way, the derivative of *g* has local extrema when $x = 0$ and $x = 3$.

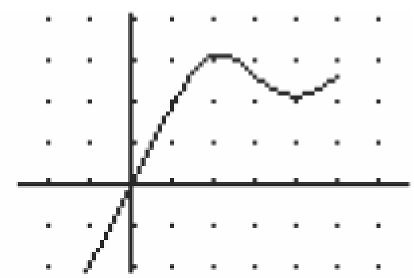
It may be an interesting exercise to use technology to draw the graph of $g$. Since the graph of $f$ consists of a circle and two line segments, we can actually find the formula for $f(t)$:

$$f(t)=\begin{cases}\sqrt{4-t^2}, & -2\le t\le 2\\ -1+|t-3|, & 2<t\le 5\end{cases}$$

Using the definition of $g(x)$, we now know:

$$g(x)=\begin{cases}\int_0^x \sqrt{4-t^2}\,dt, & -2\le x\le 2\\ \pi+\int_2^x\left(-1+|t-3|\right)dt, & 2<x\le 5\end{cases}$$

and we can produce the graph of $g$:



Please note that, as would be expected on an AP Exam, all the justifications in our solution use the graph of $f$ that was given, and not the graph of $g$ that we drew.

**Euler's Method**

Up to this point, we have looked at solving differential equations geometrically (slope fields) and analytically (separating variables). Now we wish to look at a numerical approach to approximating a solution, called Euler's Method. This topic is a BC topic, but not an AB topic.

Few differential equations can actually be solved analytically. Leonhard Euler, an eighteenth century Swiss mathematician, introduced this simple numerical approximation to the solution to a differential equation; his method is actually just an arithmetical way of following the lines in a slope field the way your eyes fill in the gaps visually.

For a specific example, consider the differential equation $dy/dx = 2xy$ with initial condition $y(1) = 1$. A word about notation in the following discussion: we will let $Y(x)$ denote the true solution, and $y(x)$ denote our piecewise linear approximation to $Y$. From the differential equation we see that the slope of the tangent line at (1, 1) is 2(1)(1) = 2. The slope of the tangent line is 2—i.e., $\Delta y/\Delta x = 2$ —so an increment of $\Delta x = 0.1$ will result in an increment of $\Delta y = 2\Delta x = 0.2$. Following the tangent line brings us to a new point $(x + \Delta x, y + \Delta y) = (1.1, 1.2)$. We now leave the line tangent to $Y$, and use the slope given by the slope field at this new point. The value of the slope field at this point is 2(1.1)(1.2) = 2.64. Now when we follow the line with slope 2.64, an increment of $\Delta x = 0.1$ will result in an increment of $\Delta y = 2.64 \cdot \Delta x = 0.264$, and we arrive at our third point, (1.2, 1.464). This process continues in the same pattern, constructing a piecewise linear continuous function.

The chart below shows the steps from point to point:

| Point | $x$ | $y$ | Slope = $2xy$ | $\Delta x$ | $\Delta y$ | $(x + \Delta x, y + \Delta y)$ |
|---|---|---|---|---|---|---|
| (1, 1) | 1 | 1 | 2 | 0.1 | 0.2 | (1.1, 1.2) |
| (1.1, 1.2) | 1.1 | 1.2 | 2.64 | 0.1 | 0.264 | (1.2, 1.464) |
| (1.2, 1.464) | 1.2 | 1.464 | 3.5136 | 0.1 | 0.35136 | (1.3, 1.81536) |

The solution to the differential equation is:

$$Y = e^{x^2-1},$$

so the true value of the function at $x = 1.3$ is $e^{(1.69 - 1)} \approx 1.9937$. The approximations become less accurate as we move farther away from the point (1, 1), the only point that we know to be on the solution curve. Notice that the true value is larger than the approximation. This is to be expected because the actual solution is concave up (can you check this using the second derivative?), and so our linear approximations lie below the curve.

The approximation can be made more accurate by choosing a smaller step size. For example, if $\Delta x = 0.05$, then after 6 steps we arrive at the point (1.3, 1.8955), cutting the error in half. In fact, the error is roughly proportional to step size, so if we let $\Delta x = 0.025$, then we expect to cut the error in about half again. This observation can lead to an improved use of Euler's Method. There are also other numerical techniques available to solve differential equations, just as there are different numerical methods of approximating a definite integral: Riemann sums, Trapezoidal Rule, Simpson's Rule, etc. Euler's Method corresponds to Riemann sums. We will stay within the bounds of the AP Calculus Course Description here though, and not go further in this discussion.

**Conclusion**

The goals of the AP Calculus course that relate to differential equations and slope fields are for the student to:

- Become familiar with the terminology of differential equations
- Recognize what is meant by a solution to a differential equation
- Use differential equations in modeling applications
- Understand the relationship between a slope field and a solution curve for the differential equation

The specific tasks that are expected of a student include these:

- Verify whether or not a given function is a solution to a differential equation
- Manually construct a portion of a slope field for a given differential equation
- Choose from among many differential equations which one is associated with a given slope field
- Choose from among many slope fields which one is associated with a given differential equation
- Recognize exponential growth or decay, the governing differential equation $dy/dx = kt$, and its solution $y = Ae^{kt}$
- Solve a given separable differential equation

## Solutions to Numbered Examples

**Example 1:**

$dy/dx = px^{p-1}$, and $y = x^p$. We need to solve $3x(px^{p-1}) = x^p$, or $3px^p = x^p$; thus $p = 1/3$.

**Example 2:**

$\frac{dy}{dx} = \frac{B}{2\sqrt{Bx+3}} = \frac{B}{2y}$. So we want $\frac{B}{2y} = \frac{5}{y}$, and $B = 10$.

**Example 3:**

$dy/dx = 7Ae^{Ax}$, and $d^2y/dx^2 = 7A^2e^{Ax}$.

Set $7A^2e^{Ax} = 9y = 63e^{Ax}$. Thus $A^2 = 9$, and $A$ can be either 3 or -3.

**Example 4:**

1. We would expect to see zero slopes for $\sin(x + y) = 0$, along the line $y = -x$. We would expect to see zero slopes for $\sin(x - y)$ along the line $y = x$.

2. Slope field A corresponds to $dy/dx = \sin(x + y)$. Slope field B corresponds to $dy/dx = \sin(x - y)$.

3. Graph B shows stripes with slope 1 because $dy/dx = \sin(x - y)$ is constant when $x - y$ is constant along lines with slope 1. Graph A shows stripes of slope -1 because $dy/dx = \sin(x + y)$ is constant along lines with slope -1.

4. In graph A, look for the horizontal slopes along the line $x + y = \pi$, and in graph B look for the horizontal slopes along the line $x - y = \pi$. Both lines cross the $x$-axis at $x = \pi$.

**Example 5:**

1. Slope field A shows slopes that vary vertically but are constant horizontally; that is, they depend on $y$ but not $x$. Slope field B shows slopes that vary horizontally but are constant vertically; that is, they depend on $x$ but not $y$.

2. Slope field B corresponds to $dy/dx = \sin(x^3)$, and slope field A corresponds to $dy/dx = \sin(y^3)$. The slope field for $dy/dx = \sin(x^3)$ (graph B) is symmetric with respect to the $y$-axis. Since $\sin(x^3)$ is an odd function, the slopes on either side of the $y$-axis are opposites of each other and appear as reflections. For the same reason, the slope field for $dy/dx = \sin(y^3)$ (graph A) is symmetric with respect to the $x$-axis.

**Example 6:**

1. Slope field A corresponds to $dy/dx = \cos(xy)$, and slope field B corresponds to $dy/dx = \sin(xy)$. There are various ways of telling them apart; the next two answers give two of the simplest.

2. Slopes along either axis should be 0 (as in slope field B) for $dy/dx = \sin(xy)$ and should be 1 (as in slope field A) for $dy/dx = \cos(xy)$.

3. Since sine is an odd function, the slope field for $dy/dx = \sin(xy)$ should be symmetric with respect to both axes (see the argument in Example 5). Slope field B shows this symmetry. Since cosine is an even function, the slopes for $dy/dx = \cos(xy)$ should look the same in all four quadrants, as in slope field A.

**Example 7:**

Note which slope fields show zero slopes along the $y$-axis and which slope fields show positive slopes in the first and third quadrants. The upper row has $dy/dx = x/y$ on the left and $dy/dx = y/x$ on the right; the bottom row has $dy/dx = -y/x$ on the left and $dy/dx = -x/y$ on the right.

**Example 8:**

Step 1: Separate variables and antidifferentiate both sides with respect to *x*:

$$\int 2y\frac{dy}{dx}dx = \int x^2 \sin(x^3)dx$$

Step 2: Use the substitution $u = x^3$ on the right side. On the left-hand side we use the substitution $y = y(x)$, obtaining:

$$\int 2y dy = \frac{1}{3}\int \sin(u)du$$

Antidifferentiating, we get:

$$y^2 = -\frac{1}{3}\cos(u) + C$$

Back-substituting yields:

$$y^2 = -\frac{1}{3}\cos(x^3) + C$$

Step 3: Use the initial condition to find the constant of integration:

$$7^2 = -\frac{1}{3}\cos(2^3) + C;\ C \approx 48.9515$$

Now you can solve for *y* and answer the question that was posed above:

$$y^2 = -\frac{1}{3}\cos(x^3) + 48.9515$$

Taking square roots of both sides we get:

$$y(x) = \sqrt{-\frac{1}{3}\cos(x^3) + 48.9515} \quad \text{or} \quad y(x) = -\sqrt{-\frac{1}{3}\cos(x^3) + 48.9515}$$

We must choose the positive square root in order to satisfy the initial condition. Thus:

$$y(3) \approx \sqrt{-\frac{1}{3}\cos(27) + 48.9515} \approx 7.00349$$

**Example 9:**

First we rewrite the equation by factoring out the coefficient of $y$; we get $y' = 3(y + 2)$. Next we let $w = y + 2$, so that $dw/dt = dy/dt$. The substitution was chosen so that missing = missing = $3(y + 2) = 3w$. We now have $dw/dt = 3w$, a familiar differential equation; we recognize the solution is $w = Ae^{3t}$. Now back-substitute to get rid of $w$, and we have y + 2 = $Ae^{3t}$. The general solution is $y = Ae^{3t} - 2$; using $y(0) = 7$, we get the particular solution to be . $y = 9e^{3t} - 2$.

We have presented this method of solving the equation to help students recognize that the derivative is just a translation of $3y$, and so the solution should just be a translation of the familiar exponential solution. Note the horizontal asymptote $w = 0$ (as $t \to -\infty$) corresponds to the horizontal asymptote $y = -2$ of the solution.

Also we factored out the 3 to begin with because that is also the way we recommend starting the more usual separation of variables argument. The next step is to divide by $y + 2$, so there is no chain rule involved in the antidifferentiation that leads to the logarithm and so no constant to be dealt with when exponentiating.

**Example 10:**

The FTC lets us immediately write:

$$y(5) = y(3) + \int_3^5 \sin(x^2)dx = 7 + \int_3^5 \sin(x^2)dx\,.$$

We must use a calculator to numerically integrate, getting $y(5) = 6.754$.

Note we have used the FTC in a slightly altered form, namely:

$$f(b) = f(a) + \int_a^b f'(t)dt\,.$$

This way of stating it only depends on a simple arithmetic change from the more usual way, but leads to a deeper understanding of the definite integral as an accumulator of change. The functional value at $b$ is the functional value at $a$ plus how much the function has changed.

## Solving Separable Differential Equations: Antidifferentiation and Domain Are Both Needed!

David O. Lomen
University of Arizona

People often think that to find solutions of differential equations, you simply find an antiderivative and then use an initial condition to evaluate the constant. While this gives a start to finding solutions of initial value problems, consideration must also be given to the domain of your final result. Sometimes these considerations are obvious, as in AB6 from the 2000 AP Exam, whose solution is given below.

**Example 1**

The solution of

$$\frac{dy}{dx} = \frac{3x^2}{e^{2y}}, \quad y(0) = \frac{1}{2}$$

is obtained by separating variables and finding an antiderivative as $e^{2y}/2 = x^3 + C$, or, as this requires that $x^3 + C$ must always be positive, $y = (1/2)\ln(2x^3 + 2C)$. Choosing $C = e/2$ allows the initial condition to be satisfied, and we have the solution of this initial value problem. The requirement that $2x^3 + e > 0$, or equivalently, $x > -(e/2)^{1/3}$, is a natural condition to have the logarithm function defined, so it includes the initial value and avoids the singularity.

However, finding solutions of initial value problems for separable differential equations need not always be as straightforward, as we see in our following four examples.

**Example 2**

As a first such example, consider the initial value problem

$$\frac{dy}{dx} = \frac{1}{x}, \quad y(-1) = 2.$$

All antiderivatives may be written as

$$y(x) = \ln|-x| + C, \qquad (1)$$

and if $C = 2$ , the initial condition is satisfied. However, even though this function satisfies the differential equation and initial condition, it is NOT the solution of this differential equation.

The solution is

$$y(x) = \ln(-x) + 2\ ,\ -\infty < x < 0 \qquad (2)$$

The reason for this is that the usual definition of a solution of a differential equation is that of a differentiable function on an open interval that contains the initial $x$-value. Notice that the function in (1) is also defined for $x > 0$, while our solution must be continuous on an open interval containing the initial value at $x = -1$. Thus the domain of our solution cannot contain $x = 0$, or positive values, but must include $x = -1$. This requires the domain to be the open interval $-\infty < x < 0$.

**Example 3**

As a second example where we need to be careful, consider the initial value problem

$$dy/dx = (1/3)y^{-2},\ y(1) = 1.$$

Separating variables gives $3y^2 dy/dx = 1$, and finding antiderivatives gives

$$y^3 = x + C\ . \qquad (3)$$

Choosing $C = 0$ satisfies the initial condition. We usually like to find explicit solutions, so solving for $y(x)$ gives

$$y(x) = x^{1/3}\ . \qquad (4)$$

We notice that this function does not have a derivative at $x = 0$, so we must add the condition $x > 0$ to our equation in (4) in order for it to include $x = 1$ and be a solution of our original initial value problem. Thus the domain for our solution in (4) is $0 < x < \infty$. If our initial condition was $y(-1) = -1$, our solution would be (4) with the domain $-\infty < x < 0$.

**Example 4**

For another example where we need to limit the domain of our solution, consider

$$\frac{dy}{dx} = \sqrt{1-y^2}, \quad y(1) = 0.$$

If we separate variables and find antiderivatives, we obtain

$$\arcsin(y) = x + C. \tag{5}$$

Choosing $C = -1$ allows the initial condition to be satisfied. If we find an explicit solution by solving for $y$, we find

$$y(x) = \sin(x-1), \tag{6}$$

and this presents a problem. From our original differential equation, and the slope field in Figure 1 below, we see that our solution should never have a negative value for its derivative, whereas our solution in (6) is oscillatory. The problem occurred when we solved equation (5) for $y(x)$. In (6) we need to limit the domain of our solution to an interval of length that includes our initial value and on which the initial sine graph is increasing.

Thus the correct solution to our problem is

$$y(x) = \sin(x-1), \ -\pi/2 < x - 1 < \pi/2$$

Notice that this domain may also be written as $1 - \pi/2 < x - 1 < 1 + \pi/2$.

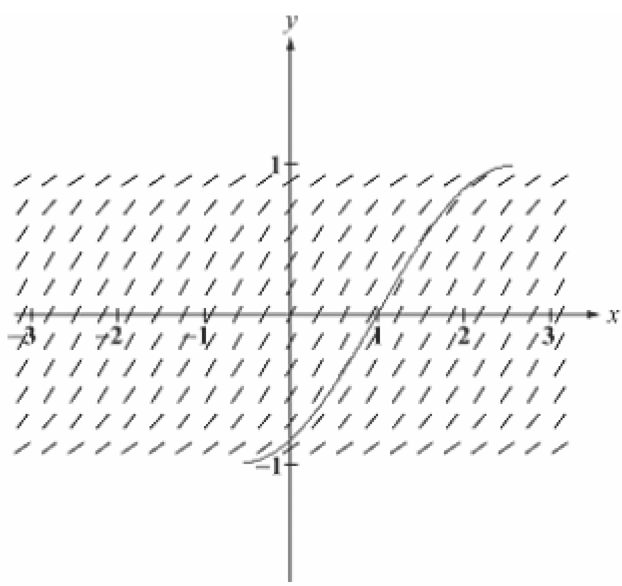



Figure 1

**Example 5**

As a final example to illustrate the need to limit the domain of an antiderivative in order to have it be a solution of a differential equation, consider

$$\frac{dy}{dx} = (y+2)^{\frac{3}{2}}, \;\; y(0) = -1.$$

Separating variables and finding antiderivatives gives

$$-2(y+2)^{-1/2} = x + C, \qquad (7)$$

so the initial condition is satisfied if . Solving (7) for $y(x)$ gives

$$y(x) = -2 + \frac{4}{(x-2)^2}. \qquad (8)$$

We notice that (8) is valid for all $x$ not equal to 2, and that this solution will have negative slopes for $x > 2$, while our original differential equation requires non-negative slopes everywhere. (A look at the slope field in Figure 2 below also demonstrates this fact.)

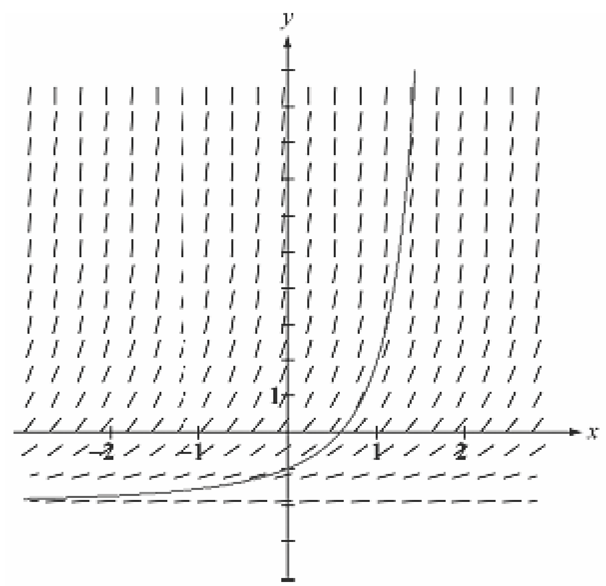



Figure 2

We got into this trouble when we squared both sides of (7) to obtain our explicit solution. Because the left side of (7) is always negative, we must limit the right side to $x + C < 0$. Thus we need to add the condition to our solution in (8).

Notice that we could not consider $x > 2$ in any case, since the domain must be an open interval containing our initial value $x = 0$, and not include $x = 2$, where the solution is not defined. This example shows that we may not extend a solution across a discontinuity, even if the resulting function formally satisfies the differential equation on the other side of the discontinuity.

**Conclusion**

So what can we conclude from these examples?

1. The process of obtaining an explicit solution from an implicit solution may result in an incorrect solution of our initial condition.

2. There may be values of x where the derivative of the explicit solution does not exist, even though it formally satisfies the differential equation.

To avoid these mistakes, we can always check to make sure that the explicit solution is such that:

- For all parts of the domain, the derivative of the explicit solution does not contradict the original differential equation. (Comparing the slope field for the differential equation with the graph of the explicit solution will display any differences.)
- Its derivative exists for all values in its domain.

For more information, see sections 1.1 and 2.4 of Lomen and Lovelock, *Differential Equations: Data, Models, and Graphics*, John Wiley and Sons, 1999.

The contributions of Larry Riddle, Ben Klein, and David Bressoud to this article were greatly appreciated.

## Graphing Calculator Programs

### Slope Field Program for the TI-83

```
8→L:12→W
FnOff
(Ymax-Ymin)/L→V
(Xmax-Xmin)/W→H
Ymin+V/3→Y
For(R,1,L)
Xmin+H/3→X
For(C,1,W)
Y1→M
⁻M*H/3+Y→S
M*H/3+Y→T
If abs(T-S)>V
Then
Y+V/3→T
Y-V/3→S
(T-Y)/M+X→Q
(S-Y)/M+X→P
Else
X-H/3→P
X+H/3→Q
End
Y→U:Line(P,S,Q,T):U→Y
X+H→X
End
Y+V→Y
End
```

**Euler's Method Program for the TI-83**

```
ClrHome
Disp "INITIAL CONDITION"
Input "XØ = ",A
Input "YØ = ",B
Disp "FINAL X ":Input "X= ",C
Disp "STEP SIZE?"
Input "H = ",H
If (abs((C-A)/(C-A))≠(abs(H/H)
⁻H→H
⁻int((⁻(C-A)/H)→N
N+1→dim(L1)
N+1→dim(L2)
N+1→dim(L3)
A→X:B→Y
Disp "SHOW STEPS?"
Disp "Ø=NO, 1=YES"
Input Z
For(I,1,N+1,1)
If Z
Then
Disp [[X,Y]]
Pause
End
X→L1(I)
Y→L2(I)
Y1→L3(I)
Y+H*Y1→Y
X+H→X
End
PlotsOff
FnOff
Plot1(xyLine,L1,L2,⁺)
PlotsOn 1
ZoomStat
```

**Documentation for the TI-83 EULER Program**

The EULER program uses Euler's Method to approximate the solution to a differential equation with an initial condition.

To use the program, follow these steps:

1. Store the slope function *dy/dx* in the Y1 function variable, using X for the independent variable and Y for the dependent variable.

2. Check that you don't need any of the data in the system list variables, L1, L2, or L3. The program uses these variables for its calculations and will overwrite anything stored there. If you do want to keep data stored there, store it to other list variables.

3. Run the program. It first prompts you for an initial condition, X0 and Y0. Enter these.

4. The program then prompts for a destination X. Enter the desired value.

5. The program then prompts for a step size, H. This corresponds to the usual $\Delta x$ in Euler's Method. Enter the desired value.

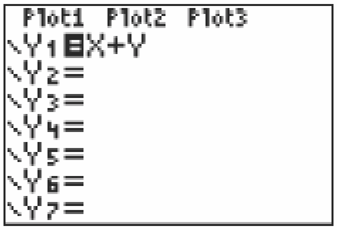
6. The program then asks if you want to see each step during the calculation. Enter 1 if you want to see the steps; enter 0 if you don't. In either case, you'll be able to see the list of *x*s, *y*s, and slopes when the program is finished.

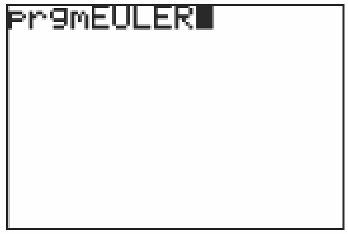
7. If you answered 1 (yes) at step 6, press Enter after each step is shown to go on to the next step. You may wish to discuss with students how the next value is calculated and ask them to calculate it before showing it to them.

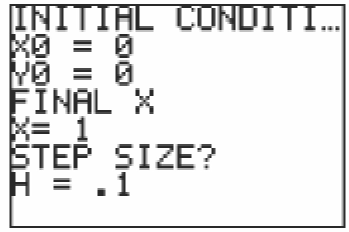
8. Once the program finishes, it displays a graph of L2 versus L1. L2 contains the *y*s and L1 the *x*s calculated during the steps of Euler's Method. You can trace on the graph to see the values.

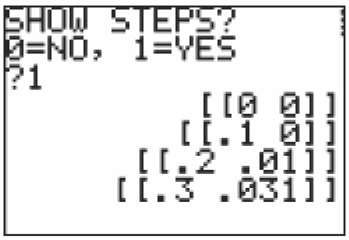
9. You can also go to the List editor (STAT EDIT) and see the lists L1, L2, and L3. L3 contains the slopes calculated at the *x* and *y* in L1 and L2. This can be useful if you are checking students' work on a test or quiz.

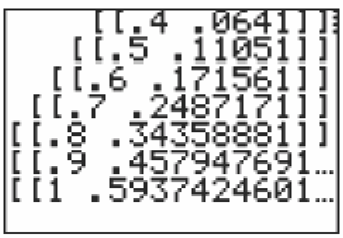
Here are screen shots illustrating these steps:

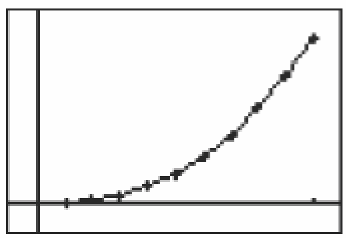


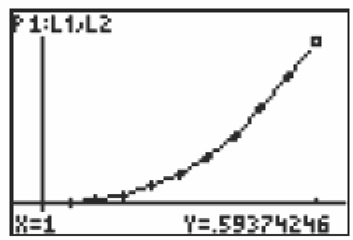



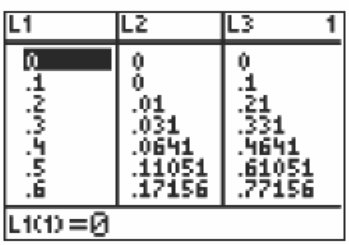





**Differential Equations—What's the Problem?**

**I. A Survey of Multiple-Choice AP Exam Questions**

| Reference | Year | Question # | % Correct | Type of Differential Equation |
|---|---|---|---|---|
| 1 | 1985 | BC 33 | 48 | Analytical with initial condition |
| 2 | 1985 | BC 44 | 52 | Slope and a point on solution curve |
| 3 | 1988 | BC 39 | 43 | Analytical with initial condition |
| 4 | 1988 | BC 43 | 44 | Rate of growth and decay |
| 5 | 1993 | AB 33 | 14 | Analytical with boundary condition |
| 6 | 1993 | AB 42 | 30 | Rate of growth and decay |
| 7 | 1993 | BC 13 | 34 | Analytical |
| 8 | 1993 | BC 38 | 63 | Rate of growth and decay |
| 9 | 1997 | AB 11 | 68 | Graphical, given $f'$ find $f$ |
| 10 | 1997 | BC 83 | 48 | Analytical with boundary condition |
| 11 | 1998 | AB 21 | 31 | Analytical |
| 12 | 1998 | AB 84 | 42 | Rate of growth and decay |
| 13 | 1998 | BC 8 | 55 | Analytical with boundary condition |
| 14 | 1998 | BC 24 | 38 | Interpret a slope field |
| 15 | 1998 | BC 26 | 20 | Logistic |

**Answers:**

| | | | | | |
|---|---|---|---|---|---|
| 1. C | 2. A | 3. C | 4. A | 5. B | 6. B |
| 7. C | 8. C | 9. E | 10. E | 11. B | 12. A |
| 13. B | 14. C | 15. E | | | |

**A Sampling of Multiple-Choice Questions—Differential Equations**

**1. 1985 BC 33**

If $dy/dt = -2y$ and if $y = 1$ when $t = 0$, what is the value of $t$ for which $y = 1/2$?

(A) $-\frac{\ln 2}{2}$ (B) $-\frac{1}{4}$ (C) $\frac{\ln 2}{2}$ (D) $\frac{\sqrt{2}}{2}$ (E) $\ln 2$

**2. 1985 BC 44**

At each point on a certain curve, the slope of the curve is $3x^2y$. If the curve contains the point (0,8), then its equation is

(A) $y = 8e^{x^3}$ (B) $y = x^3 + 8$ (C) $y = e^{x^3} + 7$

(D) $y = \ln(x+1) + 8$ (E) $y^2 = x^3 + 8$

**3. 1988 BC 39**

If $dy/dx = y \cdot \sec^2 x$ and $y = 5$ when $x = 0$, then $y =$

(A) $e^{\tan x} + 4$ (B) $e^{\tan x} + 5$ (C) $5e^{\tan x}$

(D) $\tan x + 5$ (E) $\tan x + 5e^x$

**4. 1988 BC 43**

Bacteria in a certain culture increase at a rate proportional to the number present. If the number of bacteria doubles in three hours, in how many hours will the number of bacteria triple?

(A) $\frac{3\ln 3}{\ln 2}$ (B) $\frac{2\ln 3}{\ln 2}$ (C) $\frac{\ln 3}{\ln 2}$ (D) $\ln\left(\frac{27}{2}\right)$ (E) $\ln\left(\frac{9}{2}\right)$

**5. 1993 AB 33**

If $dy/dx = 2y^2$ and if $y = -1$ when $x = 1$, then when $x = 2$, $y =$

(A) –2/3 (B) –1/3 (C) 0 (D) 1/3 (E) 2/3

**6. 1993 AB 42**

A puppy weighs 2.0 pounds at birth and 3.5 pounds two months later. If the weight of the puppy during its first 6 months is increasing at a rate proportional to its weight, then how much will the puppy weigh when it is 3 months old?

(A) 4.2 pounds (B) 4.6 pounds (C) 4.8 pounds

(D) 5.6 pounds (E) 6.5 pounds

**7. 1993 BC 13**

If $dy/dx = x^2y$, then $y$ could be

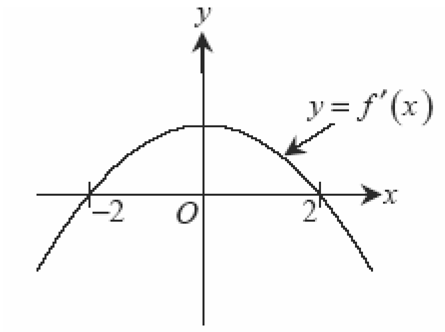
(A) $3\ln\left(\frac{x}{3}\right)$ (B) $e^{\frac{x^3}{3}}+7$ (C) $2e^{\frac{x^3}{3}}$ (D) $3e^{2x}$ (E) $\frac{x^3}{3}+1$

**8. 1993 BC 38**

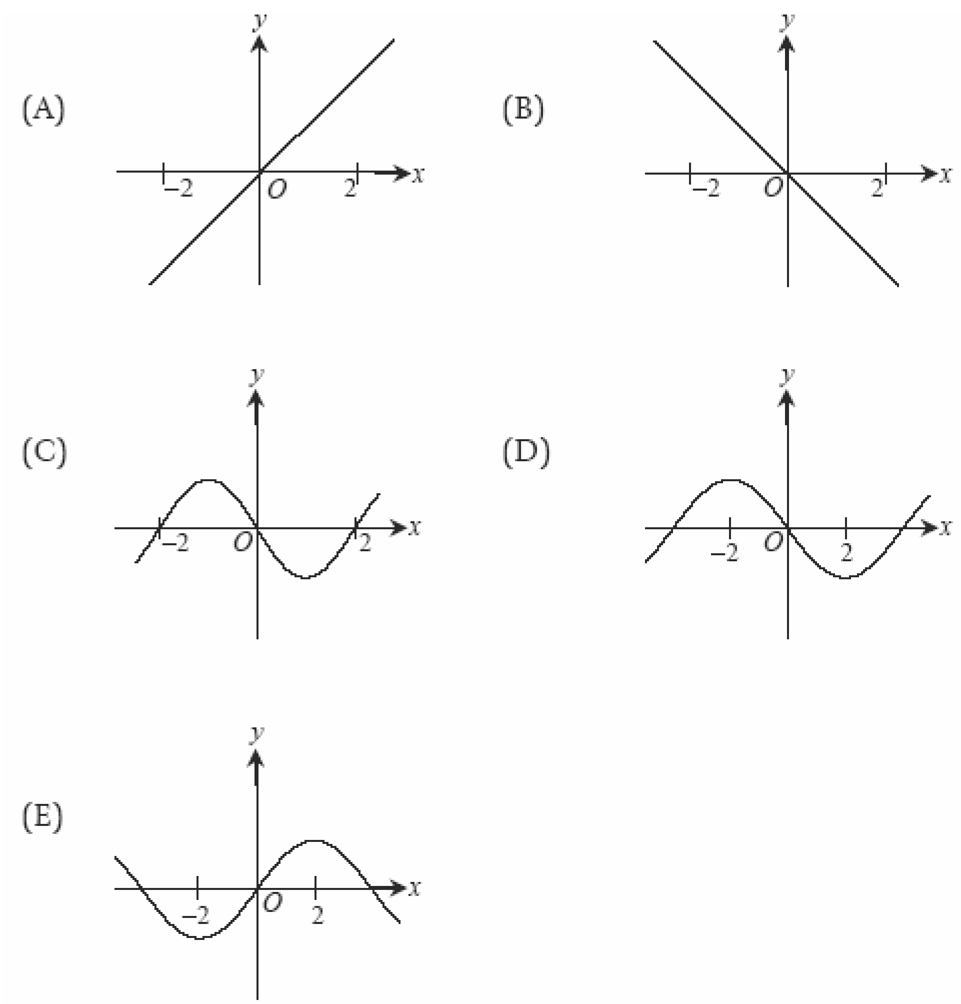
During a certain epidemic, the number of people that are infected at any time increases at a rate proportional to the number of people that are infected at that time. If 1,000 people are infected when the epidemic is first discovered, and 1,200 are infected 7 days later, how many people are infected 12 days after the epidemic is first discovered?

(A) 343 (B) 1,343 (C) 1,367 (D) 1,400 (E) 2,057

**9. 1997 AB 11**




The graph of the derivative of $f$ is shown in the figure above. Which of the following could be the graph of $f$?



**10. 1997 BC 83 (calculator allowed)**

If $dy/dx = (1 + \ln x)y$ and if $y = 1$ when $x = 1$, then $y =$

(A) $e^{\frac{x^2-1}{x^2}}$ (B) $1 + \ln x$ (C) $\ln x$ (D) $e^{2x + x\ln x - 2}$ (E) $e^{x\ln x}$

**11. 1998 AB 21**

If $dy/dt = ky$ and $k$ is a nonzero constant, then $y$ could be

(A) $2e^{kty}$ (B) $2e^{kt}$ (C) $e^{kt} + 3$ (D) $kty + 5$ (E) $\frac{1}{2}ky^2 + \frac{1}{2}$

**12. 1998 AB 84 (calculator allowed)**

Population $y$ grows according to the equation $dy/dt = ky$, where $k$ is a constant and $t$ is measured in years. If the population doubles every 10 years, then the value of $k$ is

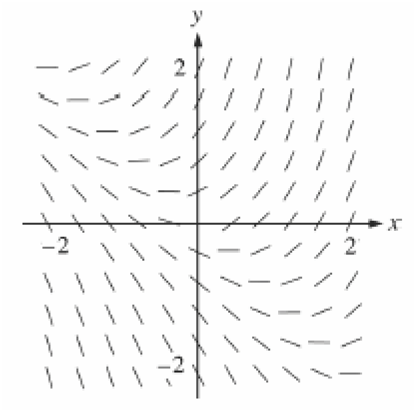
(A) 0.069 (B) 0.200 (C) 0.301 (D) 3.322 (E) 5.000

**13. 1988 BC 8**

If $dy/dx = \sin x \cos^2 x$ and if $y = 0$ when $x = \pi/2$, what is the value of $y$ when $x = 0$?

(A) –1 (B) –1/3 (C) 0 (D) 1/3 (E) 1

**14. 1998 BC 24**




Shown above is a slope field for which of the following differential equations?

(A) $\frac{dy}{dx} = 1 + x$ (B) $\frac{dy}{dx} = x^2$ (C) $\frac{dy}{dx} = x + y$ (D) $\frac{dy}{dx} = \frac{x}{y}$ (E) $\frac{dy}{dx} = \ln y$

**15. 1998 BC 26**

The population *P(t)* of a species satisfies the logistic differential equation,

$$\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$$

where the initial population $P(0) = 3{,}000$ and $t$ is the time in years. What is $\lim_{t\to\infty} P(t)$ ?

(A) 2,500 (B) 3,000 (C) 4,200 (D) 5,000 (E) 10,000

**Multiple-Choice Question Solutions—Differential Equations**

(Note: these solutions provide one way the question can be answered. There are other correct methods of solution.)

**1. 1985 BC 33**

$$\int \frac{dy}{y} = \int -2\,dt$$
$$\ln|y| = -2t + C$$
$$y = Ae^{-2t}$$
$$1 = Ae^{0}$$
$$1 = A$$

$$y = e^{-2t}$$
$$\tfrac{1}{2} = e^{-2t}$$
$$\ln\left(\tfrac{1}{2}\right) = \ln e^{-2t}$$
$$\ln\left(\tfrac{1}{2}\right) = -2t$$

$$t = -\tfrac{1}{2}\ln\left(\tfrac{1}{2}\right)$$
$$= -\tfrac{1}{2}(-\ln 2)$$
$$= \frac{\ln 2}{2}$$

Choice (C)

**2. 1985 BC 44**

$$\frac{dy}{dx} = 3x^2 y$$
$$\int \frac{dy}{y} = \int 3x^2\,dx$$
$$\ln|y| = x^3 + C$$

The point (0, 8) is on the curve

$$\ln 8 = C$$
$$\ln|y| = x^3 + \ln 8$$

$$e^{\ln|y|} = e^{x^3 + \ln 8}$$
$$|y| = e^{x^3} \cdot e^{\ln 8}$$
$$y = 8e^{x^3}$$

Choice (A)

**3. 1988 BC 39**

$$\int \frac{dy}{y} = \int \sec^2 x\,dx$$
$$\ln|y| = \tan x + C$$
$$y = Ae^{\tan x}$$

If $y = 5$ when $x = 0$

$$5 = Ae^{\tan 0}$$
$$5 = A$$
$$y = 5e^{\tan x}$$

Choice (C)

**4. 1988 BC 43**

$$\frac{db}{dt} = kb$$
$$\int \frac{db}{b} = \int k\,dt$$
$$\ln b = kt + C$$
$$b = Ae^{kt}$$
$$b(0) = A,\ b(3) = 2A$$

Find $t$ where $b(t) = 3A$

$$2A = Ae^{k\cdot 3}$$
$$2 = e^{3k}$$
$$\ln 2 = 3k$$
$$\frac{\ln 2}{3} = k$$

$$b(t) = Ae^{\frac{\ln 2}{3}t}$$
$$3A = Ae^{\frac{\ln(2)}{3}t}$$
$$3 = e^{\frac{\ln(2)}{3}t}$$
$$\ln 3 = \frac{\ln 2}{3}t$$
$$t = \frac{3\ln 3}{\ln 2}$$

Choice (A)

**5. 1993 AB 33**

$$\int \frac{dy}{y^2} = \int 2\,dx$$
$$-\frac{1}{y} = 2x + C$$

If $y = -1$ when $x = 1$

$$1 = 2 + C,\ C = -1$$
$$-\frac{1}{y} = 2x - 1$$

$$y = \frac{-1}{2x-1}$$
$$y(2) = -\frac{1}{3}$$

Choice (B)

**6. 1993 AB 42**

$$w(0) = 2,\ w(2) = 3.5$$
$$\frac{dw}{dt} = kw \text{ for } 0 \le t \le 6$$
$$\int \frac{dw}{w} = \int k\,dt$$
$$\ln w = kt + C$$

$$w = Ae^{kt}$$
$$2 = Ae^{k\cdot 0},\ \therefore A = 2$$
$$w = 2e^{kt}$$
$$3.5 = 2e^{2k}$$

$$\tfrac{7}{4} = e^{2k}$$
$$\tfrac{1}{2}\ln\tfrac{7}{4} = k$$
$$w = 2e^{\left(\frac{1}{2}\ln\frac{7}{4}\right)t} = 2\cdot\left(\tfrac{7}{4}\right)^{\frac{t}{2}}$$
$$w(3) = 2\cdot\left(\tfrac{7}{4}\right)^{\frac{3}{2}} \approx 4.63 \text{ lbs}$$

Choice (B)

**7. 1993 BC 13**

$$\int \frac{dy}{y} = \int x^2\,dx$$
$$\ln|y| = \frac{x^3}{3} + C$$

then $e^{\ln|y|} = e^{\frac{x^3}{3}+C}$

$$|y| = e^{\frac{x^3}{3}} \cdot e^C$$

So $y = Ae^{\frac{x^3}{3}}$, where $A = y(0)$

Choice (C)

**8. 1993 BC 38**

Let $p$ = the number of people infected at time $t$.

$$\frac{dp}{dt} = kp$$
$$\int \frac{dp}{p} = \int k\,dt$$
$$\ln p = kt + C$$
$$p = Ae^{kt}$$

$$p(0) = A = 1000$$
$$p = 1000e^{kt}$$
$$p(7) = 1200 = 1000e^{k\cdot 7}$$
$$\tfrac{6}{5} = e^{7k}$$
$$\ln\tfrac{6}{5} = 7k$$

$$\tfrac{1}{7}\ln\tfrac{6}{5} = k$$
$$p(t) = 1000e^{\left(\frac{1}{7}\ln\frac{6}{5}\right)t} = 1000\left(\tfrac{6}{5}\right)^{\frac{t}{7}}$$
$$p(12) = 1000\left(\tfrac{6}{5}\right)^{\frac{12}{7}} \approx 1366.908$$

or 1,367 people Choice (C)

**9. 1997 AB 11**

Based on a number line analysis:

Choice (E)

| | | | | | |
|---|---|---|---|---|---|
| $f$ | dec | | inc | | dec |
| $f'$ | $-----$ | $-2$ | $+++++$ | $2$ | $-----$ |

**10. 1997 BC 83 (calculator allowed)**

Note: This question requires a student either to know $\int \ln x\, dx$ or to be able to use integration by parts to find it.

$$\int \frac{dy}{y} = \int (1+\ln x)\,dx$$
$$\ln|y| = x + x\ln x - x + C$$
$$\ln|y| = x\ln x + C$$

$$\ln|1| = 1\cdot\ln(1) + C$$
$$0 = C$$
$$\ln y = x\ln x$$
$$y = e^{x\ln x}$$

Choice (E)

**11. 1998 AB 21**

$$\int \frac{dy}{y} = \int k\,dt$$
$$\ln|y| = kt + C$$
$$y = Ae^{kt} \text{ and } A \text{ "could be" } 2$$

Choice (B)

**12. 1998 AB 84 (calculator allowed)**

$y = y(0)e^{kt}$ and if the population doubles every 10 years

$2y(0) = y(0)e^{k \cdot 10}$

$2 = e^{10k}$

$\ln(2) = 10k$ or $k = \ln(2)/10 \approx 0.069$ Choice (A)

**13. 1988 BC 8**

$$\int dy = \int \sin x \cos^2 x \, dx \qquad 0 = -\frac{\cos^3\left(\frac{\pi}{2}\right)}{3} + C \qquad y(0) = -\frac{1}{3}$$

$$y = -\frac{\cos^3 x}{3} + C \qquad 0 = C \qquad y = -\frac{\cos^3 x}{3}$$

Choice (B)

**14. 1998 BC 24**

There are numerous ways to approach this problem. One popular student method is the process of elimination:
(A) $dy/dx = 1 + x$. $dy/dx = 0$ when $x = -1$. This is not true for this slope field.

(B) $dy/dx = x^2$. $dy/dx \geq 0$. Some slopes are negative in this slope field.

(C) $dy/dx = x + y$. $dy/dx = 0$ when $y = -x$. This does match the slope field.

(D) $dy/dx = x/y$. This would have undefined slopes for $y = 0$, the $x$-axis.

(E) $dy/dx = \ln y$. Slopes exist only for $y > 0$.

Therefore, the answer must be Choice (C).

Another approach is to realize that if $dy/dx$ is a function of x only, then the slope field looks like columns of parallel segments. If $dy/dx$ is a function of y only, then the slope field looks like rows of parallel segments. Since this is not a feature of the slope field, (A), (B), and (E) are eliminated at a glance. (D) is eliminated because slopes in quadrants I and III would be positive.

**15. 1998 BC 26**

$\frac{dP}{dt} = P\left(\frac{10,000 - P}{5000}\right) = 0$ when P= 0 or P= 10,000 (the carrying capacity).

$\lim_{t \to \infty} P(t) = 10,000$ (It reaches no change at its carrying capacity.) Choice (E)

**A Sampling of Free-Response Questions—Differential Equations**

Please note that the free-response questions and scoring guidelines from recent years can be downloaded from AP Central. Sets of free-response questions and official solutions for 1969-1997 are available for sale in the College Board Store.

apcentral.collegeboard.com/calculusab
apcentral.collegeboard.com/calculusbc
store.collegeboard.com

**2001 AB 6**

The function *f* is differentiable for all real numbers. The point (3, 1/4) is on the graph of $y = f(x)$, and the slope at each point $(x,y)$ on the graph is given by $dy/dx = y^2(6 - 2x)$.

(a) Find $d^2y/dx^2$ and evaluate it at the point (3, 1/4).

(b) Find $y = f(x)$ by solving the differential equation $dy/dx = y^2(6 - 2x)$ with the initial condition $f(3) = 1/4$.

**2000 AB 6 (no calculator)**

Consider the differential equation:

$$\frac{dy}{dx} = \frac{3x^2}{e^{2y}}$$

(a) Find a solution $y = f(x)$ to the differential equation satisfying $f(0) = 1/2$.

(b) Find the domain and range of the function *f* found in part (a).

**1993 AB 6**

Let $P(t)$ represent the number of wolves in a population at time *t* years, when $t \geq 0$. The population $P(t)$ is increasing at a rate directly proportional to $800 - P(t)$, where the constant of proportionality is k.

(a) If $P(0) = 500$, find in $P(t)$ terms of *t* and *k*.
(b) If $P(2) = 700$, find *k*.
(c) Find $\lim_{t \to \infty} P(t)$.

**1992 AB 6**

At time $t$, $t \geq 0$, the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius.

At $t = 0$, the radius of the sphere is 1 and at $t = 15$, the radius is 2. (The volume $V$ of a sphere with radius $r$ is $V = (4/3)\pi r^3$.)

(a) Find the radius of the sphere as a function of $t$.

(b) At what time $t$ will the volume of the sphere be 27 times its volume at $t = 0$?

**1991 BC 6**

A certain rumor spreads through a community at the rate $dy/dt = 2y(1 - y)$, where $y$ is the proportion of the population that has heard the rumor at time $t$.

(a) What proportion of the population has heard the rumor when it is spreading the fastest?
(b) If at time $t = 0$ ten percent of the people have heard the rumor, find $y$ as a function of $t$.
(c) At what time $t$ is the rumor spreading the fastest?

**1989 AB 6**

Oil is being pumped continuously from a certain oil well at a rate proportional to the amount of oil left in the well; that is, $dy/dt = ky$, where $y$ is the amount of oil left in the well at any time $t$. Initially there were 1,000,000 gallons of oil in the well, and 6 years later there were 500,000 gallons remaining. It will no longer be profitable to pump oil when there are fewer than 50,000 gallons remaining.

(a) Write an equation for $y$, the amount of oil remaining in the well at any time $t$.
(b) At what rate is the amount of oil in the well decreasing when there are 600,000 gallons of oil remaining?
(c) In order not to lose money, at what time $t$ should oil no longer be pumped from the well?

**1987 BC 1**

At any time $t \geq 0$, in days, the rate of growth of a bacteria population is given by $y' = ky$, where $k$ is a constant and $y$ is the number of bacteria present. The initial population is 1,000 and the population triples during the first 5 days.

(a) Write an expression for $y$ at any time $t \geq 0$.

(b) By what factor will the population have increased in the first 10 days?

(c) At what time $t$, in days, will the population have increased by a factor of 6?

**1985 BC 4**

Given the differential equation $dy/dx = -xy/\ln y$, $y > 0$.

(a) Find the general solution of the differential equation.

(b) Find the solution that satisfies the condition that $y = e^2$ when $x = 0$ . Express your answer in the form $y = f(x)$.

(c) Explain why $x = 2$ is not in the domain of the solution found in part (b).

**1983 AB 5/BC 3**

At time $t = 0$, a jogger is running at a velocity of 300 meters per minute. The jogger is slowing down with a negative acceleration that is directly proportional to time $t$. This brings the jogger to a stop in 10 minutes.

(a) Write an expression for the velocity of the jogger at time $t$.

(b) What is the total distance traveled by the jogger in that 10-minute interval?

**1974 AB 7**

The rate of change in the number of bacteria in a culture is proportional to the number present. In a certain laboratory experiment, a culture had 10,000 bacteria initially, 20,000 bacteria at time $t_1$ minutes, and 100,000 bacteria at $(t_1 + 10)$ minutes.

(a) In terms of t only, find the number of bacteria in the culture at any time $t$ minutes, $t \geq 0$.

(b) How many bacteria were there after 20 minutes?

(c) How many minutes had elapsed when the 20,000 bacteria were observed?

**1971 BC 6**

Find a function $f$ that has a continuous derivative on $(0, \infty)$ and that has both of the following properties:

(i) The graph of $f$ goes through the point (1,1).

(ii) The length $L$ of the curve from (1,1) to any point $(x, f(x))$ is given by $L = \ln x + f(x) - 1$.

Note: Recall that the arc length from $(a, f(a))$ to $(b, f(b))$ is given by

$$\int_a^b \sqrt{1 + (f'(t))^2}\, dt.$$

## A CBL Exploration into Newton's Law of Heating/Cooling

**Introduction:** Newton's Law of Heating or Cooling says that the rate of change of the difference in temperature between one body and its surroundings is proportional to that temperature difference. In symbols, if $H$ is the temperature difference and $t$ the time, then $dH/dt = kH$. In this activity, you will gain some hands-on experience with this result.

**Materials:** Calculator Based Laboratory (or other data collection device), temperature probe (connect to Channel 1 on CBL), TI-83 graphing calculator with program COOLTEMP (TI-83 program on page 54), cup of ice water

**Procedure:**

1. Use the temperature probe to measure the air temperature (in degrees Centigrade) and record your result: Air Temp ___________________ °C. Store this result in the variable *A* in your calculator.

2. Immerse the temperature probe into the cup of ice water. Wait a few minutes for the temperature to stabilize.

3. Press the MODE button on the CBL so that it says DONE on the CBL display.

4. Run the COOLTEMP program.

5. Just before pressing the ENTER key to start graphing temperatures, remove the temperature probe from the cup of ice water.

6. Wait for the program to finish. Using the COOLTEMP program attached, this will take about 100 seconds.

7. Go to the STAT Edit screen and look at your data. The list L2 stores the times, and the list L4 stores the temperatures.

8. The temperatures in L4 must be temperature *differences* (between the temperature probe and the air). Subtract *A* from all the temperatures to change the temperatures in L4 into temperature differences between the probe and the air. On the TI-83, execute the command *A* – L4 -> L4.

Here are the steps to analyze the data. A sample data set appears in the Teacher Notes that follow.

**Analysis and Questions:**

1. Solve the differential equation $dH/dt = kH$ by separating variables. Don't worry about finding any of the constants just yet, but do solve for $H(t)$.

   Write down your solution here: ________________________, show your solution to your teacher, and have your teacher initial here: __________ before proceeding.

2. Your solution to the differential equation in question 1 should involve two constants. To find one of them, you'll use your temperature difference at time $t = 0$. Solve for the first constant and write your formula for $H(t)$:

   ____________________________________

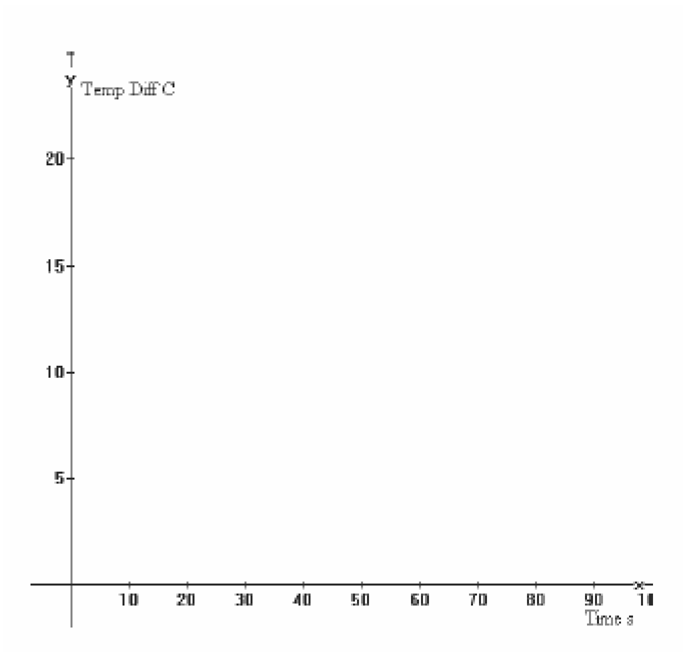
3. The next constant you'll find is $k$. To do this, you will use temperature differences at three times during the experiment and average the values for $k$ that you find at each. Select three times (one early in the experiment, one about halfway through, and one at the end). At each time, use the values of $t$ and $H$ to calculate $k$. Record them here:

| $t$ | $H$ | $k$ |
|---|---|---|
| | | |
| | | |
| | | |

4. Average the three values of k you found in step 3 and write down your formula for $H(t)$ here:

   ____________________________________

5. Store your formula for $H(t)$ (substituting $X$ for $t$ as the independent variable) into Y1 and overlay its graph on your scatter plot of temperature differences versus time. Sketch your graph on the axes provided below:




6. Your graph should have a horizontal asymptote. What is the equation for this asymptote?

________________________________________

7. Why should your graph have this for its horizontal asymptote?

____________________________________________________________________________

____________________________________________________________________________

____________________________________________________________________________

## Teacher Notes for "A CBL Exploration into Newton's Law of Heating/Cooling"

This activity can be used any time after students have studied separation of variables to solve differential equations. However, separation of variables is used only to answer Question 1.

Students must also be familiar with using logarithms to isolate a variable in an exponential equation.

You could modify the activity to record temperatures as the probe is cooling in the ice water and then repeat the entire analysis for that setting.

Instead of just averaging three values of $H$ to approximate $k$ at Questions 3 and 4, you could have students find $k$ for all the values of $H$ (except the one at $t = 0$) and average those. The calculator lets you do this with the following two commands:

1. `seq(ln(L4(X)/L4(Ø))/L2(X),X,2,99,1) → L1`
   (The `seq` command is in the LIST OPS menu.)
   This creates a list of 98 values of $k$ and stores it in L1.

2. `mean(L1) → k`

**Answers:**

The ambient air temperature for the analysis below was 16.8 °C. These answers came from the sample data below.

1. $H(t) = H_0e^{kt}$

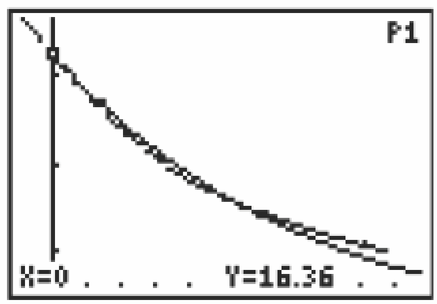
2. At $t = 0$, $H = 16.36$. So $H_0 = 16.36$, and $H(t) = 16.36e^{kt}$.

3.

| $t$ | $H$ | $k$ |
|---|---|---|
| 30 | 10.48 | –0.015 |
| 50 | 8.13 | –0.014 |
| 80 | 6 | –0.013 |

4. $k = -0.014$; $H(t) = 16.36e^{-0.014t}$

5. Here is the scatter plot, with $y = H(t)$ overlaid.




6. The line $y = 0$ is the horizontal asymptote.

7. This is because as time approaches infinity, the temperature difference between the probe and the surrounding air approaches 0.

8. The temperature difference, $H$, changes the fastest at the outset of the experiment, when the temperature difference between the probe and the air is the greatest. Newton's Law of Cooling says that the rate of change of the temperature difference is proportional to the temperature difference. When the difference is the greatest, the rate of change is the greatest.

9. As time goes on, the temperature difference decreases (in fact, as mentioned in number 7, the temperature difference approaches 0). Thus, the rate of change of the temperature difference also decreases.

## Sample Data

| Time | Temp Diff Time °C | Time | Temp Diff Time °C |
|---|---|---|---|
| | | | |
| 0 | 16.36 | 50 | 8.13 |
| 1 | 16.22 | 51 | 8.01 |
| 2 | 15.93 | 52 | 8.01 |
| 3 | 15.65 | 53 | 8.03 |
| 4 | 15.37 | 54 | 7.9 |
| 5 | 15.23 | 55 | 7.65 |
| 6 | 14.95 | 56 | 7.66 |
| 7 | 14.67 | 57 | 7.54 |
| 8 | 14.57 | 58 | 7.54 |
| 9 | 14.43 | 59 | 7.42 |
| 10 | 14.15 | 60 | 7.17 |
| 11 | 13.88 | 61 | 7.3 |
| 12 | 13.75 | 62 | 7.18 |
| 13 | 13.48 | 63 | 7.18 |
| 14 | 13.34 | 64 | 7.06 |
| 15 | 13.72 | 65 | 6.94 |
| 16 | 12.94 | 66 | 6.83 |
| 17 | 12.68 | 67 | 6.83 |
| 18 | 12.54 | 68 | 6.71 |
| 19 | 12.28 | 69 | 6.71 |
| 20 | 12.15 | 70 | 6.59 |
| 21 | 12.02 | 71 | 6.59 |
| 22 | 11.76 | 72 | 6.47 |
| 23 | 11.63 | 73 | 6.47 |
| 24 | 11.5 | 74 | 6.35 |
| 25 | 11.25 | 75 | 6.23 |
| 26 | 11.12 | 76 | 6.23 |
| 27 | 11.14 | 77 | 6.23 |
| 28 | 10.73 | 78 | 6.11 |
| 29 | 10.61 | 79 | 6 |
| 30 | 10.48 | 80 | 6 |
| 31 | 10.35 | 81 | 5.88 |
| 32 | 10.23 | 82 | 5.88 |
| 33 | 10.25 | 83 | 5.76 |
| 34 | 9.85 | 84 | 5.76 |
| 35 | 9.85 | 85 | 5.76 |
| 36 | 9.6 | 86 | 5.65 |
| 37 | 9.48 | 87 | 5.65 |
| 38 | 9.48 | 88 | 5.53 |
| 39 | 9.23 | 89 | 5.53 |
| 40 | 9.11 | 90 | 5.41 |
| 41 | 9.11 | 91 | 5.54 |
| 42 | 8.98 | 92 | 5.41 |
| 43 | 8.86 | 93 | 5.3 |
| 44 | 8.88 | 94 | 5.3 |
| 45 | 8.62 | 95 | 5.18 |
| 46 | 8.49 | 96 | 5.18 |
| 47 | 8.49 | 97 | 5.18 |
| 48 | 8.51 | 98 | 5.07 |
| 49 | 8.25 | | |

**COOLTEMP Program for TI-83**

```
PlotsOff
Func
FnOff
AxesOn
Ø→Xmin
99→Xmax
1Ø→Xscl
⁻2→Ymin
35→Ymax
1Ø→Yscl
ClrList L2,L4
ClrHome
{1,Ø}→L1
Send(L1)
{1,1,1}→L1
Send(L1)
99→dim(L4)
ClrHome
Disp "PRESS ENTER TO"
Disp "START GRAPHING"
Disp "TEMPERATURE:"
Pause
ClrDraw
Text(4,1,"TEMP(C)":Text(54,81,"T(S)")
{3,1,⁻1,Ø}→L1
Send(L1)
For(I,1,99,1)
Get(L4(I))
Pt-On(I,L4(I))
End
seq(N,N,Ø,98,1)→L2
Ø→Xmin
max(L2)→Xmax
1Ø→Xscl
Plot1(Scatter,L2,L4,·)
ZoomStat
Text(4,1,"TEMP(C)"):Text(54,81,"T(S)")
Stop
```

## Sample Calculus Reviews from the Teachers' Resources Area at AP Central

The Teachers' Resources area at AP Central hosts teacher-written reviews of resources for many AP courses. Included are over 350 reviews for Calculus AB and Calculus BC of textbooks, books, calculators, professional associations, software, Web sites, and other teaching resources. Written by veteran AP teachers and college instructors, these reviews describe the contents of a resource and also explore how to incorporate it into an AP classroom.

A sample of reviews from the Teachers' Resources area, focusing on the theme of differential equations, follows on the next pages.

To find these and other reviews in the Teachers' Resources area, log in to AP Central (apcentral.collegeboard.com) and click the button on the main navigation bar at the top of the screen marked "Teachers' Resources." On the search page, select "Calculus AB" or "Calculus BC" and the type of resource. You can also enter a keyword or words, such as "differential equations" or "slope fields."

The Teachers' Resources area is a work-in-progress, and new reviews are added on a regular basis. Please contact AP Central with review ideas or suggestions.

Located at: apps.apcentral.collegeboard.com/ResourceDetail.jsp?ResourceID=3451

| | |
|---|---|
| Title: | MathServ DE Toolkit |
| Author: | Philip Crooke from Vanderbilt University<br>Steve Tschantz from Vanderbilt University |
| Course: | Calculus AB<br>Calculus BC |
| Abstract/Summary: | The MathServ DE Toolkit is an online resource for solving differential equations. The user enters the equation, clicks, and the solution is returned. Twelve of the simplest types of differential equations are available from boxes on this page, the first being separable differential equations — the only kind required for the AP Calculus Exams. Solvers for additional types of equations are available from links on this page. One of these is a link that will give the antiderivative; another will plot the slope field, here called a tangent field. The solution to an initial value problem may be graphed on top of the slope field. In each case the user types in the problem and in a few seconds the answer is returned.<br><br>The Web site is actually an interface with Mathematica 4.0. Users do not need Mathematica on their computers in order to use the site, or even need to know how to use Mathematica. There are boxes in which the information is entered. Once the information is entered, a click of a button produces the answer or graph.<br><br>The slope field grapher could be used to produce slope fields with or without solutions that can be cut and pasted into documents. The Toolkit provides a quick way to get or check answers. It does not provide any explanation or help along the way. For that reason you may — or may not — want your students to know about it. Once they understand the basics, it may be useful for getting the symbolic manipulation or graphing done quickly. |
| Type: | Web site |
| Cost: | Free |
| AP Specific: | No |
| Link to Resource: | http://mss.math.vanderbilt.edu/~pscrooke/detoolkit.html |
| Resource Includes | Reference Material<br>Calculator/Computer Programs |
| Reviewer Name: | Lin McMullin |

Located at: apps.apcentral.collegeboard.com/ResourceDetail.jsp?ResourceID=3999

Title: *Revolutions in Differential Equations*
Author: Michael J. Kallaher, ed.
Course: Calculus AB
Calculus BC
Abstract/Summary: This publication, edited by Michael J. Kallaher, contains eight articles regarding the teaching of differential equations. They are:

1. *Modeling and Visualization in the Introductory ODE Course* by Robert L. Borelli
2. *Differential Equations in the Information Age* by William E. Boyce
3. *A Geometric Approach to Ordinary Differential Equations* by Michael Branton & Margie Hale
4. *Differential Equations on the Internet* by Kevin D. Cooper & Thomas LoFaro
5. *Data as an Essential Part of a Course in Differential Equations* by David O. Lomen
6. *Qualitative Study of Differential Equations* by Valipuram S. Manoranjan
7. *Teaching Numerical Methods in ODE Courses* by Lawrence F. Shampine & Ian Gladwell
8. *Technology in Differential Equations Courses: My Experiences, Student Reactions* by Beverly H. West

While not aimed specifically at the AP Calculus differential equations topics, there is information here on those topics. More importantly, these articles give perspectives on the teaching of differential equations and can provide AP teachers with a background that goes beyond what is necessary in the AP courses. For example, the first article talks about the use of "laboratory" experiments in teaching differential equations and gives several examples of differential equations arising from "real world" phenomena. The second article talks about the style of instruction, while the third article focuses on slope fields. The fourth article discusses the integration of instruction with Web-based materials and gives a list of differential equations Web sites. The fifth article indicates how data can be used to construct a differential equation that models the data. Article 7 examines the important issues concerning the use of numerical methods, and Article 8 examines the importance of the use of technology in the teaching of differential equations.

For the teacher having little experience with teaching differential equations, wanting “laboratory” experiences using technology for their students, or ideas for projects that students could do after the exam, this could be a valuable resource.

| | |
|---|---|
| Publication Data: | Mathematical Association of America<br>1529 Eighteenth Street, NW<br>Washington, DC 20036-1385<br>Copyright 1999<br>ISBN 0-88385-160-1 |
| Type: | Book |
| Cost: | $18.75 |
| Vendor Name: | Mathematical Association of America |
| Vendor Phone: | (800) 331-1622 |
| AP Specific: | No |
| Resource Includes | Background Material<br>Explorations<br>Project Ideas |
| Reviewer Name: | Jim Hartman |

Located at: apps.apcentral.collegeboard.com/ResourceDetail.jsp?ResourceID=11

| | |
|---|---|
| Title: | WinPlot |
| Author: | Richard Parris from Phillips Exeter Academy |
| Course: | Calculus AB<br>Calculus BC |
| Abstract/Summary: | This is one of the best graphing programs you will find anywhere, and it's free. WinPlot will graph rectangular, polar, parametric, and implicitly defined relations. Functions of two variables in several formats may be graphed in 3D. Probably the best feature is the graphing of solids of revolution and solids with regular cross sections. You can enter the equation, specify the axis for revolution, and view the solid, which can be rotated to see all sides. The original 2D graph can be overlaid on the 3D graph and it too can be rotated to show how the solid is generated. This feature alone makes WinPlot worth having, but the program also graphs slope fields, gives Riemann sums, and displays zeros, intersections, derivatives, and integrals. The final graphs may be copied and pasted into documents.<br><br>There is a separate Web site with very good Instructions for Using WinPlot written by Al Lehnen at Madison Area Technical College in Madison, Wisconsin. |
| Type: | Software |
| Cost: | Free |
| Vendor Name: | Peanut Software |
| Vendor Phone: | (603) 772-4311 |
| AP Specific: | No |
| Link to More Info: | http://math.exeter.edu/rparris/default.html |
| Link to Resource: | http://math.exeter.edu/rparris/peanut/winplotz.exe |
| Resource Includes | Calculator/Computer Programs |
| Reviewer Name: | Lin McMullin |

## Contributors

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**About the Editor**

**Susan Kornstein** is the Director of Content Development for K-12 Professional Development at the College Board. Before working for the College Board, she taught mathematics at both the college and high school level for 26 years, most recently as head of the mathematics department at Horace Mann School in New York City and as AP Calculus and AP Statistics teacher at Rye Country Day School in Rye, New York. She was involved in the Advanced Placement Program as a Reader, Table Leader, and Question Leader at the AP Calculus Reading and as a College Board consultant for AP Calculus, AP Statistics, and Mathematics Vertical Teams. Susan was awarded the Presidential Award for Excellence in Mathematics and Science Teaching, the MAA Edyth May Sliffe Award for Distinguished High School Mathematics Teaching, and the RadioShack National Teacher Award. Susan attended New York City public schools, including the Bronx High School of Science, and has a bachelor's degree in mathematics from the City College of New York (Belden Prize in Advanced Calculus) and master's and M.Phil degrees in statistics from Yale University.

**Benita Albert** is a 35-year teacher of Advanced Placement Calculus at Oak Ridge High School in Oak Ridge, Tennessee. She has served as a College Board consultant in the Southeast Region, as well as a member of the AP Calculus Development Committee and as a Reader and Table Leader for the AP Exam grading. Her latest efforts include planning committee assignments for College Board programs, Math Vertical Teams, and Building Success in Mathematics.

**David Bressoud** is DeWitt Wallace Professor of Mathematics at Macalester College in St. Paul, Minnesota, and chair of the AP Calculus Development Committee. As a Peace Corps volunteer, he taught seventh and eighth grade math and science at the Clare Hall School in Antigua, West Indies, 1971-73. He was a professor at Penn State from 1977 until 1994, and in 1990-91 he taught an AP Calculus class at the State College Area High School in State College, Pennsylvania. He has authored several textbooks that use history to motivate the study of mathematics, including *Second Year Calculus from Celestial Mechanics to Special Relativity* and *A Radical Approach to Real Analysis*.

**Judy Broadwin** taught AP Calculus for 30 years at JerichoHigh School in Jericho, New York. She has been a Reader, Table Leader, and twice Exam Leader and served on the AP Calculus Development Committee that was responsible for the new Course Description introduced in 1998. Recently she was a consultant in Advanced Placement working for the College Board in New York City, where her responsibilities included planning workshops and the Siemens competition for both schools and teachers. She was the co-author of the Math Olympiads Solutions Book for AP Calculus, published from 1982 to 2001.

**Ray Cannon** received three hours of college credit after taking the AP Calculus Exam in 1958. He started grading the exams 20 years later and has since served in various roles including Chief Reader (1992-95), member of the AP Calculus Development Committee, and College Board Workshop consultant. He is presently a professor at Baylor University.

**Thomas Dick** is a professor of mathematics at Oregon State University, where he is faculty director of OSU's Mathematics Learning Center. He works extensively with both preservice and inservice mathematics teachers at all levels. He served on the AP Calculus Development Committee 1995-2002 (chair, 1998-2002) and is currently co-chair of the College Board/Mathematical Association of America's Committee on Mutual Concerns.

**Mark Howell** is a long-time teacher of Advanced Placement Calculus and Computer Science at his alma mater, Gonzaga College High School in Washington, D.C. Mark has served the Advanced Placement community for many years, as a consultant, Reader of AP Exams, Table Leader, and Question Leader. He served on the AP Calculus Development Committee. His efforts supporting the use of technology to enhance the teaching and learning of mathematics have taken him throughout the continental U.S., as well as Puerto Rico, Hawaii, Singapore, Australia, and Thailand.

**Dan Kennedy** has taught AP Calculus and other subjects at the Baylor School in Chattanooga, Tennessee, for 30 years. He chaired the AP Calculus Development Committee from 1990 to 1994 and has conducted more than 50 workshops and institutes for high school teachers. He has co-authored five textbooks on subjects ranging from Algebra I to calculus and is a frequent contributor of material to the AP Central Web site.

**David Lomen** has taught mathematics for 45 years, and is currently a University Distinguished Professor at the University of Arizona. He has served as an AP Reader and workshop consultant for many years, and is currently a member of the AP Calculus Development Committee. He has authored or co-authored 39 research articles on applied mathematics, 17 educational articles, and 6 textbooks (on algebra, calculus, differential equations, and linear algebra). He has also given over three dozen workshops on teaching calculus and teaching with technology, and was the chairman of the National Advisory Committee for Systemic Teaching Excellence Projects to the state of Montana and the commonwealth of Puerto Rico.