



AP[®] Calculus:

Reasoning from Tabular Data

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Curriculum Module

AP Calculus

Reasoning from Tabular Data

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Reasoning from tabular data has been a recurring topic in recent AP Calculus Exam questions, and one that has proved to be very challenging for students. This lesson urges you to make reasoning from tabular data a recurring theme throughout your AP Calculus course. It reviews how to estimate rates of change, net change, and average value from values of functions given in tables, and helps you help students learn to determine which formula or method to use in various situations and applications.

Introducing the Topic

Often in real-world applications of Calculus such as population, temperature, fluid flow, and motion, data are available, but a closed form function is not known. For this reason students must learn to work from data to approximate quantities such as the rate of change of the function, the net change of the function, and the average value of the function.

I introduce this topic during the unit on derivatives by having students use data to estimate the rate of change, the slope, and the derivative. During the unit on definite integrals and numerical integration, I have students use data to estimate the total change and the average value of a function. When I review the students for the AP Exam, I use problems like the ones below that ask the students to use data to estimate the value of both the derivative and the definite integral. Reasoning from tabular data should be a recurring topic that is incorporated throughout the course everywhere that it is appropriate.

Example 1

Let $y(t)$ represent the population of a town over a 20-year period, where y is a differentiable function of t . The table below shows the population recorded at selected times.

t (yrs)	0	4	10	13	20
$y(t)$ (people)	2500	2724	3108	3697	4283

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1. Use data from the table to find an approximation for $y'(12)$, and explain the meaning of $y'(12)$ in terms of the population of the town. Show the computations that lead to your answer.

Solution

Since $t = 12$ is not one of the times given in the table, we should approximate the derivative by using a difference quotient with the best (closest) data available. Because 12 lies between 10 and 13, the best approximation for $y'(12)$ is found by:

$$y'(12) \approx \frac{y(13) - y(10)}{13 - 10} = \frac{3697 - 3108}{3} = 196.333 \text{ people per year.}$$

When $t = 12$ years, the population of the town is increasing at a rate of approximately 196.333 people per year.

Notice that since the difference quotient is equal to the change in population divided by the change in years, the units for the answer are people per year. Notice, too, that the difference quotient is also the average rate of change of the population function from time $t = 10$ to $t = 13$ years.

Note: It is not necessary to simplify answers on the AP free-response questions, but the computations must be shown. Answers must be given to three decimal places unless otherwise instructed.

2. Use data from the table and a trapezoidal approximation with four subintervals to approximate the average population of the town over the 20-year period. Show the computations that lead to your answer.

Solution

The average value of $y(t)$ from $t = 0$ to $t = 20$ years is given by:

$$\begin{aligned} & \frac{1}{20} \int_0^{20} y(t) dt \\ & \approx \frac{1}{20} \left(\frac{1}{2}(4)(2500 + 2724) + \frac{1}{2}(6)(2724 + 3108) + \frac{1}{2}(3)(3108 + 3697) + \frac{1}{2}(7)(3697 + 4283) \right) \\ & = 3304.075 \text{ people.} \end{aligned}$$

The average population over the 20-year period is approximately 3304.075 people.

Notice that $y(t)$ is measured in people, and t is measured in years so the units of:

$$\frac{1}{20} \int_0^{20} y(t) dt \text{ are } \left(\frac{1}{\text{yrs}} \right) (\text{people})(\text{yrs}) = \text{people}.$$

This makes sense because the average population should have the number of people as the units.

Notice also that the function given in this problem represents the *population* of the town. Therefore the *average population* is the *average value of the population function*, which is found by using the formula:

$$\text{Average Value of } f \text{ on } [a, b] = \frac{1}{b-a} \int_a^b f(x) dx.$$

Example 2

Sometimes we are given data about a function and also a model for the function. When this occurs, we can compare the results given by the data and by the model to help determine how good the model is and to help check whether we have made an error in our computations.

The rate at which water flows into a tank, in gallons per hour, is given by a positive continuous function R of time t . The table below shows the rate at selected values of t for a 12-hour period.

t (hrs)	0	2	4	6	8	10	12
$R(t)$ (gal/hr)	12.5	13.4	13.9	14.3	14.6	14.8	14.7

1. Use a midpoint Riemann sum with three subintervals to approximate:

$$\int_0^{12} R(t) dt,$$

and explain the meaning of this definite integral in terms of the water flow, using correct units. Show the computations that lead to your answer.

Solution

Since $\int_0^{12} R(t) dt \approx 4(R(2) + R(6) + R(10)) = 4(13.4 + 14.3 + 14.8) = 170$ gallons, approximately 170 gallons of water flowed into the tank between $t = 0$ and $t = 12$ hours.

Notice that the integral $\int_0^{12} R(t) dt$ does not have a coefficient of $\frac{1}{12}$ so the integral gives the *total change* or *net change*, not the *average value of the rate of change*. The integral of a rate of change is the total change or net change. Another way to recognize that the value of the integral is the total change is to notice the units in the

problem. Since $R(t)$ is measured in gallons per hour, and t is measured in hours, the integral has units of:

$$\left(\frac{\text{gal}}{\text{hr}}\right)(\text{hr}) = \text{gal}.$$

2. A model for the rate of water flow is given by the function:

$$P(t) = \frac{1}{60}(750 + 24t - t^2),$$

where the positive rate P is measured in gallons per hour and the time t is measured in hours. Use $P(t)$ to find the average rate of water flow during the 12-hour time period. Indicate units of measure.

Solution

The average rate of water flow is given by $\frac{1}{12} \int_0^{12} P(t) dt = \frac{1}{12}(169.2) = 14.1$ gal/hr.

Notice that the value of $\int_0^{12} P(t) dt$ is 169.2 gallons, which is very close to the answer to (1).

Notice also that since the function given in this problem represents the rate of water flow, then the average rate of water flow is just the average value of the given function, which is found by the formula

$$\text{Average Value of } f \text{ on } [a, b] = \frac{1}{b-a} \int_a^b f(x) dx.$$

In this case, the given function is a rate of change function, and the average rate of change is the average value of the given rate of change function. If the function given in the problem had represented the amount of water in the tank, then the average rate of change would have been found by the formula

$$\text{Average Rate of Change of } f \text{ on } [a, b] = \frac{f(b) - f(a)}{b - a}.$$

As in part (1), the units are helpful in determining the meaning of the answer.

$$\frac{1}{12} \int_0^{12} P(t) dt \text{ has units of } \left(\frac{1}{\text{hr}}\right)\left(\frac{\text{gal}}{\text{hr}}\right)(\text{hr}) = \frac{\text{gal}}{\text{hr}}$$

which are the units for the average value of the rate of change function.

Students often confuse the average rate of change formula and the average value formula. They should be careful to note whether the given function represents an amount, such as the amount of water in a tank, or a rate of change of an amount.

Example 3

Particle A moves along a horizontal line with a velocity $v_A(t)$, where $v_A(t)$ is a positive continuous function of t . The time t is measured in seconds, and the velocity is measured in cm/sec. The velocity $v_A(t)$ of the particle at selected times is given in the table below.

t (sec)	0	2	5	7	10
$v_A(t)$ (cm/sec)	1.7	6.8	7.4	15.6	24.9

- Use data from the table to approximate the distance traveled by particle A over the interval $0 \leq t \leq 10$ seconds by using a right Riemann sum with four subintervals. Show the computations that lead to your answer, and indicate units of measure.

Solution

$$\text{Distance} \approx (2)(6.8) + (3)(7.4) + (2)(15.6) + (3)(24.9) = 141.7 \text{ cm.}$$

Notice that this quantity approximates the integral $\int_0^{10} v_A(t) dt$ and that the units are:

$$(\text{sec}) \left(\frac{\text{cm}}{\text{sec}} \right) = \text{cm}.$$

- Particle B moves along the same line with an acceleration of $a_B(t) = 2t - 7$ cm/sec². At time $t = 1$ second, the velocity of particle B is 13 cm/sec. Which particle is traveling faster at time $t = 5$ seconds? Explain your answer.

Solution

Let $v_B(t)$ be the velocity of particle B at time t . Then:

$$v_B(t) = \int (2t - 7) dt = t^2 - 7t + C.$$

At $t = 1$, we have $13 = 1 - 7 + C$, so that $C = 19$ and $v_B(t) = t^2 - 7t + 19$.

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Hence, $v_B(5) = 9 > 7.4 = v_A(5)$, and we conclude that particle B is traveling faster at time $t = 5$ seconds.

Worksheet and AP Examination Questions

The following worksheet on Reasoning from Tabular Data includes additional examples and problems for students, as well as solutions for the instructor.

Students have been asked to work with tabular data on the AP Calculus AB and BC Examinations numerous times over the past few years. Tabular data was given in the following problems, available at AP Central® (apcentral.collegeboard.com) at The AP Calculus AB Exam page or at The AP Calculus BC Exam page. From the AP Calculus AB Course Home Page, select Exam Information: The AP Calculus AB Exam; from the AP Calculus BC Course Home Page, select Exam Information: The AP Calculus BC Exam.

2003 AB-3

2003 Form B AB-3/BC-3

2004 Form B AB-3/BC-3

2005 AB-3/BC-3

2006 AB-4/BC-4

2006 Form B AB-6

2007 AB-5/BC-5

Tabular data also was used in the 2003 BC Exam Multiple Choice Question 25, available for purchase at AP Central.

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Worksheet: Reasoning from Tabular Data

Use your graphing calculator, and give decimal answers correct to three decimal places.

1. Let $y(t)$ represent the temperature of a pie that has been removed from a 450°F oven and left to cool in a room with a temperature of 72°F , where y is a differentiable function of t . The table below shows the temperature recorded every five minutes.

t (min)	0	5	10	15	20	25	30
$y(t)$ ($^\circ\text{F}$)	450	388	338	292	257	226	200

- a. Use data from the table to find an approximation for $y'(18)$, and explain the meaning of $y'(18)$ in terms of the temperature of the pie. Show the computations that lead to your answer, and indicate units of measure.

- b. Use data from the table to find the value of $\int_{10}^{25} y'(t) dt$, and explain the meaning of $\int_{10}^{25} y'(t) dt$ in terms of the temperature of the pie. Indicate units of measure.

- c. A model for the temperature of the pie is given by the function:

$$W(t) = 72 + 380e^{-0.036t},$$

where t is measured in minutes and $W(t)$ is measured in degrees Fahrenheit ($^\circ\text{F}$).

Use the model to find the value of $W'(18)$. Indicate units of measure.

- d. Use the model given in part (c) to find the time at which the temperature of the pie is 300°F .

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2. Let $y(t)$ represent the population of the town of Sugar Mill over a 10-year period, where y is a differentiable function of t . The table below shows the population recorded every two years.

t (yrs)	0	2	4	6	8	10
y (people)	2500	2912	3360	3815	4330	4875

- a. Use data from the table to find an approximation for $y'(7)$, and explain the meaning of $y'(7)$ in terms of the population of Sugar Mill. Show the computations that lead to your answer.
- b. Use data from the table to approximate the average population of Sugar Mill over the time interval $0 \leq t \leq 10$ by using a left Riemann sum with five equal subintervals. Show the computations that lead to your answer.
- c. A model for the population of another town, Pine Grove, over the same 10-year period is given by the function:

$$P(t) = (2t + 50)^2,$$

where t is measured in years and $P(t)$ is measured in people. Use the model to find the value of $P'(7)$.

- d. Use the model given in part (c) to find the value of:

$$\frac{1}{10} \int_0^{10} P(t) dt$$

Explain the meaning of this integral expression in terms of the population of Pine Grove.

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3. A bowl of soup is placed on the kitchen counter to cool. Let $T(x)$ represent the temperature of the soup at time x , where T is a differentiable function of x . The temperature of the soup at selected times is given in the table below.

x (min)	0	4	7	12
$T(x)$ (°F)	108	101	99	95

- a. Use data from the table to find:

$$\int_0^{12} T'(x) dx$$

Explain the meaning of this definite integral in terms of the temperature of the soup.

- b. Use data from the table to find the average rate of change of $T(x)$ over the time interval $x = 4$ to $x = 7$.

- c. Explain the meaning of:

$$\frac{1}{12} \int_0^{12} T(x) dx$$

in terms of the temperature of the soup, and approximate the value of this integral expression by using a trapezoidal sum with three subintervals.

4. The rate at which water is being pumped into a tank is given by the continuous, increasing function $R(t)$. A table of selected values of $R(t)$, for the time interval $0 \leq t \leq 20$ minutes, is shown below.

t (min)	0	4	9	17	20
$R(t)$ (gal/min)	25	28	33	42	46

- a. Use a right Riemann sum with four subintervals to approximate the value of:

$$\int_0^{20} R(t) dt.$$

Is your approximation greater or less than the true value? Give a reason for your answer.

- b. A model for the rate at which water is being pumped into the tank is given by the function:

$$W(t) = 25e^{0.03t},$$

where t is measured in minutes and $W(t)$ is measured in gallons per minute. Use the model to find the average rate at which water is being pumped into the tank from $t = 0$ to $t = 20$ minutes.

- c. The tank contained 100 gallons of water at time $t = 0$. Use the model given in part (b) to find the amount of water in the tank at $t = 20$ minutes.

5. Car A has positive velocity $v_A(t)$ as it travels on a straight road, where v_A is a differentiable function of t . The velocity is recorded for selected values over the time interval $0 \leq t \leq 10$ seconds, as shown in the table below.

t (sec)	0	2	5	7	10
$v_A(t)$ (ft/sec)	0	9	36	61	115

- a. Use data from the table to approximate the acceleration of Car A at $t = 8$ seconds. Indicate units of measure.
- b. Use data from the table to approximate the distance traveled by Car A over the interval $0 \leq t \leq 10$ seconds by using a trapezoidal sum with four subintervals. Show the computations that lead to your answer, and indicate units of measure.
- c. Car B travels along the same road with an acceleration of $a_B(t) = 2t + 2$ ft/sec². At time $t = 3$ seconds, the velocity of Car B is 11 ft/sec. Which car is traveling faster at time $t = 7$ seconds? Explain your answer.

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6. A particle moves along a horizontal line with a positive velocity $v(t)$, where v is a differentiable function of t . The time t is measured in seconds, and the velocity is measured in cm/sec. The velocity of the particle at selected times is given in the table below.

t (sec)	0	2	4	6	8	10	12
$v(t)$ (cm/sec)	37	17	5	1	6	17	38

- Based on the values in the table, what is the smallest number of times at which the velocity of the particle could equal 20 cm/sec in the open interval $0 < t < 12$ seconds? Justify your answer.
- Based on the values in the table, what is the smallest number of times at which the acceleration of the particle could equal zero in the open interval $0 < t < 12$ seconds? Justify your answer.
- Find the average acceleration of the particle over the time interval $8 \leq t \leq 10$ seconds. Show the computations that lead to your answer, and indicate units of measure.
- Use a midpoint Riemann sum with three subintervals of equal length and values from the table to approximate:

$$\int_0^{12} v(t) dt$$

Show the computations that lead to your answer. Using correct units, explain the meaning of this definite integral in terms of the particle's motion.

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Worksheet: Reasoning from Tabular Data— Solutions

1. a. $y'(18) \approx \frac{257 - 292}{20 - 15} = -7^\circ\text{F} / \text{min}$. When $t = 18$ minutes, the temperature of the pie is decreasing at a rate of approximately 7°F per minute.
- b. $\int_{10}^{25} y'(t) dt = y(25) - y(10) = -112^\circ\text{F}$. From $t = 10$ minutes to $t = 25$ minutes, the temperature of the pie dropped 112°F .
- c. $W'(18) = -7.156^\circ\text{F}$ per minute.
- d. $W(t) = 300$ when $t = 14.190$ minutes.

2. a. $y'(7) \approx \frac{4330 - 3815}{8 - 6} = 257.5$. When $t = 7$ years, the population of Sugar Mill is increasing at a rate of approximately 257.5 people per year.

- b. Average population:

$$= \frac{1}{10} \int_0^{10} y(t) dt \approx \frac{1}{10} (2(2500 + 2912 + 3360 + 3815 + 4330)) = 3383.4$$

so the average population over the 10-year period was approximately 3383.4 people.

- c. $P'(7) = 256$ people per year.

- d. $\frac{1}{10} \int_0^{10} P(t) dt = 3633.333$ people. This means that the average population of Pine Grove over the 10-year period was approximately 3633.333 people.

3. a. $\int_0^{12} T'(x) dx = T(12) - T(0) = -13^\circ\text{F}$.

From $x = 0$ to $x = 12$ minutes, the temperature of the soup dropped 13°F .

- b. Average rate of change = $\frac{T(7) - T(4)}{7 - 4} = \frac{99 - 101}{3} = -\frac{2}{3}^\circ\text{F} / \text{min}$.

- c. $\frac{1}{12} \int_0^{12} T(x) dx$ represents the average temperature of the soup over the 12-minute period and is approximately equal to:

$$\frac{1}{12} \left(\frac{1}{2}(4)(108+101) + \frac{1}{2}(3)(101+99) + \frac{1}{2}(5)(99+95) \right) = 100.25^\circ\text{F}.$$

4. a. $\int_0^{20} R(t) dt \approx (4)(28) + (5)(33) + (8)(42) + (3)(46) = 751$ gallons

Since R is positive, this is an estimate of the amount of water pumped into the tank during the 20-minute period. Since R increases on $0 < t < 20$, the right Riemann sum approximation of 751 gallons is greater than:

$$\int_0^{20} R(t) dt.$$

b. Average rate = $\frac{1}{20} \int_0^{20} W(t) dt = 34.255$ gal/min.

c. $100 + \int_0^{20} W(t) dt = 785.099$ gallons.

5. a. Let $a_A(t)$ be the acceleration of Car A at time t . Then:

$$a_A(8) \approx \frac{v_A(10) - v_A(7)}{10 - 7} = \frac{115 - 61}{3} = 18 \frac{\text{ft/sec}}{\text{sec}} = 18 \frac{\text{ft}}{\text{sec}^2}.$$

- b. Distance =

$$\int_0^{10} v_A(t) dt \approx \frac{1}{2}(2)(0+9) + \frac{1}{2}(3)(9+36) + \frac{1}{2}(2)(36+61) + \frac{1}{2}(3)(61+115) = 437.5 \text{ ft}$$

- c. Let $v_B(t)$ be the velocity of Car B at time t . Then:

$$v_B(t) = \int (2t + 2) dt = t^2 + 2t + C$$

At $t = 3$, we have $11 = 9 + 6 + C$, so that $C = -4$ and $v_B(t) = t^2 + 2t - 4$. Hence, $v_B(7) = 59 < 61 = v_A(7)$. We conclude that Car A is traveling faster at time $t = 7$ seconds.

6. a. Two times. Since $v(0) = 37 > 20 > 17 = v(2)$ and $v(10) = 17 < 20 < 38 = v(12)$ and v is continuous, by the Intermediate Value Theorem, there exists a value t_1 in $(0, 2)$ such that $v(t_1) = 20$ and a value t_2 in $(10, 12)$ such that $v(t_2) = 20$.
(We know that v is differentiable on its domain so it is also continuous there.)

- b. One time. Since $v(2) = v(10) = 17$ and v is continuous and differentiable, by the Mean Value Theorem, there exists a value t in $(2, 10)$ such that:

$$a(t) = v'(t) = \frac{17 - 17}{10 - 2} = 0.$$

Or, since $v(t_1) = v(t_2) = 20$ and $t_1 < t_2$ for the t_1 and t_2 found in part (a), there exists a value t in (t_1, t_2) such that:

$$a(t) = v'(t) = \frac{20 - 20}{t_2 - t_1} = 0.$$

- c. Average acceleration = $\frac{v(10) - v(8)}{10 - 8} = \frac{17 - 6}{2} = 5.5 \frac{\text{cm/sec}}{\text{sec}} = 5.5 \frac{\text{cm}}{\text{sec}^2}$.

- d. $\int_0^{12} v(t) dt \approx 4(v(2) + v(6) + v(10)) = 4(17 + 1 + 17) = 140 \text{ cm}$. Since the velocity is positive, $\int_0^{12} v(t) dt$ represents the distance, in cm, traveled by the particle from $t = 0$ to $t = 12$ seconds.

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