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AP® Calculus:

Fundamental Theorem of Calculus

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Curriculum Module

AP Calculus

Fundamental Theorem of Calculus

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This lesson features an activity in which students who do not already know the First Fundamental Theorem of Calculus discover the Second Fundamental Theorem of Calculus through a graphical exploration. The lesson also includes several worksheets designed to reinforce and review both parts of the Fundamental Theorem of Calculus. The Fundamental Theorem of Calculus is an important and unifying theme for the AP Calculus course.

Introducing the Topic:

The Fundamental Theorem of Calculus makes the connection between differential Calculus and integral Calculus in the statement:

If $F(x) = \int_a^x f(t)\,dt$, where f is continuous, then $\frac{dF(x)}{dx} = f(x)$,

and defines the definite integral as accumulated (or total) change in the statement:

$\int_a^b f(x)\,dx = F(b) - F(a)$, where F is an antiderivative of f.

Although the first statement above is often identified in calculus textbooks as the Second Fundamental Theorem of Calculus and the second statement as the First Fundamental Theorem of Calculus, I introduce these statements to my students in the order in which they appear above. This lesson consists of five worksheets for students, two on the Second Fundamental Theorem of Calculus and three on the First Fundamental Theorem. The first of these worksheets is a classroom activity designed to help students discover the Second Fundamental Theorem through a graphical exploration. I use it after students understand the definite integral as a signed area and as a limit of Riemann sums, but before they learn to use antiderivatives to evaluate definite integrals.

My students discover the Second Fundamental Theorem of Calculus during two class days of intensive work on Worksheet 1, Examples 1–4 (and similar examples), and then a third day of work on Example 5 (and similar examples) during which students and teacher work together to discover how the Chain Rule is incorporated into the Fundamental Theorem. Worksheet 2 allows students to apply their newfound knowledge of the Second Fundamental Theorem of Calculus. I use Worksheet 2 as an immediate follow-up to Worksheet 1, but note that Worksheet 2 also could be used to enable students who already know the First Fundamental Theorem of Calculus to discover the Second Fundamental Theorem.

We then derive the First Fundamental Theorem of Calculus, as shown at the top of Worksheet 3, and thoroughly discuss its meaning in terms of physical applications, especially distance and velocity. Students can then practice their understanding of the definite integral of a rate of change as total change using Worksheet 3. It is only after completing Worksheet 3 that my calculus classes study antiderivatives and use the First Fundamental Theorem of Calculus to evaluate definite integrals. Worksheets 4 and 5 can be used to introduce or review various applications and interpretations of the First Fundamental Theorem of Calculus. (Worksheet 4 also provides review of the Second Fundamental Theorem of Calculus.) They can be used for review either at the end of the unit on this topic or just before students take the AP Exam.

Worksheets and AP Examination Questions

Each of the following worksheets includes additional notes for the instructor, along with solutions.

On the AP exams, students must use the Fundamental Theorem of Calculus to complete both multiple choice questions (see Worksheet 4) and free response questions. During 2004-2007 the following free response questions required use of the Second Fundamental Theorem of Calculus.

2004 AB5
2005 AB4 parts c and d
2005 Form B AB4/BC4
2006 AB3
2007 AB3 part c

The following questions required use of the First Fundamental Theorem of Calculus, specifically to find total change as the definite integral of the rate of change, as in Worksheet 3.

2004 AB1/BC1 part a, AB3 part d
2004 Form B AB2 parts c and d, AB3/BC3 part a
2005 AB2, AB3/BC3 part c, and AB5/BC5 part a
2005 Form B AB2/BC2, AB3 part c
2006 AB2, AB4/BC4 parts b and c
2006 Form B AB4/BC4 parts c and d, AB6 part b
2007 AB2/BC2 parts a and c, AB5/BC5 part c
2007 Form B AB2 parts b and c

Free response questions from recent AP Calculus examinations are available at AP Central (apcentral.collegeboard.com) at The AP Calculus AB Exam page or at The AP Calculus BC Exam page. From the AP Calculus AB Course Home Page, select Exam

Information: The AP Calculus AB Exam; from the AP Calculus BC Course Home Page, select Exam Information: The AP Calculus BC Exam.

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Worksheet 1. Areas, Derivatives, and the Fundamental Theorem of Calculus (or Discovering the Second Fundamental Theorem of Calculus: A Laboratory Exercise)

Teacher Notes for Worksheet 1

I remember learning how to read: I had recognized that the letters C A T symbolically represented a picture of a feline with whiskers. And so, in keeping with the "rule of four": that functions are to be represented graphically, numerically, analytically, and verbally, the following exercise has our students "drawing pictures" on a rectangular grid in order to connect the two sides of the equation giving the Second Fundamental Theorem of Calculus; namely,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

I hope that, through these applications, our students will learn to read the statement of the Second Fundamental Theorem of Calculus.

Prior to doing the following exercises, the student should be able to:

- Sketch regions bounded by graphs of functions in the Cartesian plane and calculate areas of these regions using Euclidean geometry. For instance, given a region bounded above by the graph of $f(x) = 2x + 3$ and below by the x-axis, on $-1 \le x \le 4$, the student should be able to apply the area formula for a trapezoid to calculate the area of the given region as 30.
- Read, sketch, and calculate a definite integral as the signed area of the region(s) bounded by the graph of the integrand function, the x-axis, and the vertical lines given by the limits of integration—that is, as the sum of areas of regions above the x-axis, minus the sum of areas of regions below the x-axis. For example, given $\int_{-2}^{5}(-3)\,dx$, the student should be able to draw a region bounded above by the x-axis and below by the graph of $f(x) = -3$, on $-2 \le x \le 5$, and calculate the definite integral as the signed area of this region; namely, $\int_{-2}^{5}(-3)\,dx = -21$. Given $\int_{-2}^{5}(2-x)\,dx$, the student should be able to calculate the definite integral as $\int_{-2}^{5}(2-x)\,dx = 8 - \frac{9}{2} = \frac{7}{2}$.

- Apply the properties of integrals: $\int_a^a f(x)\,dx = 0$,

  $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$, and

  if $a \le b \le c$ then

  $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$.

Worksheet 1 allows students to discover the Second Fundamental Theorem of Calculus by leading them to establish a relationship between signed areas and the function that bounds the regions having these areas. Each of the first two pages has two examples with the same "boundary" function. What changes is the "starting" point so we can graphically study the significance of the constant of integration. Page 3 (Example 5) introduces the Chain Rule. On all three pages, you may wish to extend or enlarge the graphing grids for your students and/or encourage students to choose the scales for their axes wisely.

My students complete Worksheet 2 immediately after they complete Worksheet 1, to reinforce and test their understanding of the Second Fundamental Theorem of Calculus.

Solutions to Worksheet 1

Example 1.

(b) See table below. (d) $F_1(x) = 3x$. (e) $F_1'(x) = 3$.

x	-2	-1	0	1	2	3	General Formula
$F_1(x)$	$3(-2-0)$	$3(-1-0)$	$3(0-0)$	$3(1-0)$	$3(2-0)$	$3(3-0)$	$3(x-0)$

Example 2.

(b) See table below. (d) $F_2(x) = 3x - 3$. (e) $F_2'(x) = 3$.

x	-2	-1	0	1	2	3	General Formula
$F_2(x)$	$3(-2-1)$	$3(-1-1)$	$3(0-1)$	$3(1-1)$	$3(2-1)$	$3(3-1)$	$3(x-1)$

(f) The graphs of $F_1(x)$ and $F_2(x)$ are straight lines having the same slope. The functions $F_1(x)$ and $F_2(x)$ have the same derivative. At this point, you may wish to ask students to predict or compute the derivative of $F(x) = \int_a^x 3\,dt$, where a is any real number, obtaining $F'(x) = 3$.

Example 3.

(b) See table below. (d) $G_1(x) = 1(x-0)^2 + 0 = x^2$. (e) $G_1{}'(x) = 2x$.

x	-2	-1	0	1	2	3	General Formula
$G_1(x)$	$\frac{-4(-2-0)}{2}$	$\frac{-2(-1-0)}{2}$	$\frac{0(0-0)}{2}$	$\frac{2(1-0)}{2}$	$\frac{4(2-0)}{2}$	$\frac{6(3-0)}{2}$	$\frac{2x(x-0)}{2}$

Example 4.

(b) Note that using the formula for the (signed) area of a trapezoid throughout may lead students more directly to the general formula.

x	-2	-1	0	1	2	3	General Formula
$G_2(x)$	$\frac{(-4+2)(-2-1)}{2}$	$\frac{(-2+2)(-1-1)}{2}$	$\frac{(2)(0-1)}{2}$	$\frac{(2+2)(1-1)}{2}$	$\frac{(2+4)(2-1)}{2}$	$\frac{(2+6)(3-1)}{2}$	$\frac{(2+2x)(x-1)}{2}$

(d) $G_2(x) = 1(x-0)^2 - 1 = x^2 - 1$

(e) $G_2{}'(x) = 2x$

(f) The graphs of $G_1(x)$ and $G_2(x)$ are parabolas with the same slope at points with the same x-coordinate. The functions $G_1(x)$ and $G_2(x)$ have the same derivative. As a follow-up, you may wish to ask students to predict or compute the derivative of $G(x) = \int_a^x 2t\,dt$, where a is any real number, obtaining $G'(x) = 2x$.

Example 5.

(a) See table below. (c) $H(u)=u^2$. (d) $H'(u)=2u$.

x	-2	-1	$-\frac{1}{3}$	0	$\frac{1}{3}$	1	2	3	
u	-6	-3	-1	0	1	3	6	9	General Formula
$H(u)$	$\frac{-12(-6-0)}{2}$	$\frac{-6(-3-0)}{2}$	$\frac{-2(-1-0)}{2}$	0	$\frac{2(1-0)}{2}$	$\frac{6(3-0)}{2}$	$\frac{12(6-0)}{2}$	81	$\frac{2u(u-0)}{2}$

(e) $H(x)=(3x)^2$

(f) $H'(x)=2(3x)3=2u\dfrac{du}{dx}$

Note that differentiating $H(x)=(3x)^2$ using the Chain Rule, rather than simplifying to $H(x)=9x^2$ before differentiating, helps lead students more directly to the general formula.

Notice also that students may write the formula for *H* in terms of *x* from the table in part (a). An interesting exercise is to direct students to use the horizontal axis for both *x* and *u*, scaling so that $u=3x$. (See the chart above: $u=-6$ should correspond to $x=-2$, and $u=3$ should correspond to $x=1$, for instance.) If students then plot both the points $(u,H(u))$ and the points $(x,H(x))$, they will find that the graphs of $y=H(u)$ and $y=H(x)$ coincide, with $H(u)=u^2$ and $H(x)=(3x)^2$.

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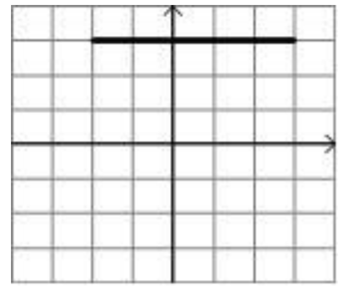
Worksheet 1. Areas, Derivatives, and the Fundamental Theorem of Calculus

Example 1.

The graph of $f(t)=3$ on $[-2,3]$ is shown at right.

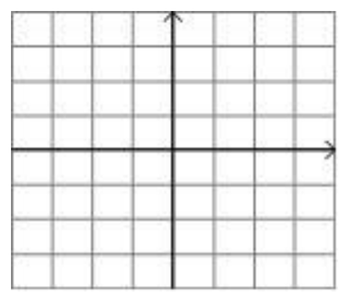


(a) Shade the region bounded by the given curve, the x-axis, and the vertical lines $x=-2$ and $x=3$.

(b) Fill in the chart by calculating the signed area $F_1(x)=\int_0^x 3\,dt$ for each value of x using the geometric formula for area of a rectangle and properties of integrals.

x	-2	-1	0	1	2	3	General Formula
$F_1(x)$							

(c) Graph the ordered pairs $(x, F_1(x))$ on the grid below.



(d) Write a linear function $F_1(x)$ that represents the graph from part (c). (It should match the formula from the chart above.)

$F_1(x) =$ ________________

(e) What is $F_1'(x)$? ________________

Example 2.

The graph of $f(t)=3$ on $[-2,3]$ is shown above.

(a) Refer to part (a) from Example 1.

(b) Fill in the chart by calculating the signed area $F_2(x)=\int_1^x 3\,dt$ for each value of x using the geometric formula for the area of a rectangle and properties of integrals.

x	-2	-1	0	1	2	3	General Formula
$F_2(x)$							

(c) Graph the ordered pairs $(x, F_2(x))$ on the grid on which you graphed $F_1(x)$.

(d) Write a linear function $F_2(x)$ that represents the graph from part (c). (It should match the formula from the chart above.)

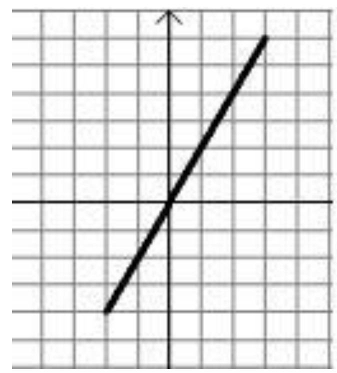
$F_2(x) =$ __________________

(e) What is $F_2'(x)$? __________________

(f) Compare $F_1(x)$ and $F_2(x)$. How are they the same?

Example 3.

The graph of $g(t) = 2t$ on $[-2,3]$ is shown at right.

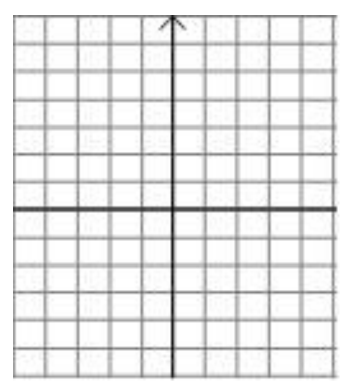


(a) Shade the region bounded by the given curve, the x-axis, and the vertical lines $x = -2$ and $x = 3$.

(b) Fill in the chart by calculating the signed area $G_1(x) = \int_0^x 2t\,dt$ for each value of x using the geometric formula for the area of a triangle and properties of integrals.

x	-2	-1	0	1	2	3	General Formula
$G_1(x)$							

(c) Graph the ordered pairs $(x, G_1(x))$ on the grid below.



(d) Write a quadratic function $G_1(x)$ that represents the graph from part (c). (It should match the formula from the chart above.)

$G_1(x) =$ __________________

(e) What is $G_1'(x)$? __________________

Example 4.

(a) Refer to part (a) from Example 3.

(b) Fill in the chart by calculating the signed area $G_2(x) = \int_1^x 2t\,dt$ for each value of x using the geometric formulas for areas of triangles and trapezoids and properties of integrals.

x	-2	-1	0	1	2	3	General Formula
$G_2(x)$							

(c) Graph the ordered pairs $(x, G_2(x))$ on the grid on which you graphed $G_1(x)$.

(d) Write a quadratic function $G_2(x)$ that represents the graph from part (c). (It should match the formula from the chart above.)

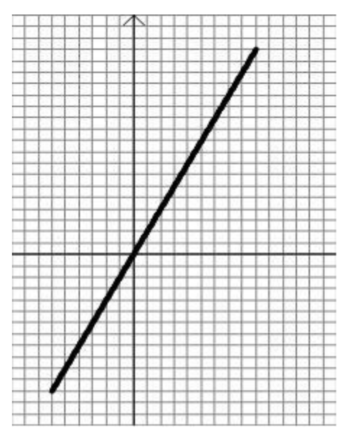
$G_2(x) =$ __________________

(e) What is $G_2{}'(x)$? __________________

(f) Compare $G_1(x)$ and $G_2(x)$. How are they the same?

Example 5. Chain Rule

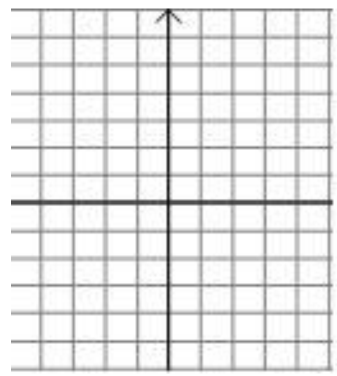
The graph of $f(t) = 2t$ on $[-6, 9]$ is shown at right.



(a) Complete the chart by filling in the remaining values of u, and then calculating the signed area $H(u) = \int_0^{u=3x} 2t\,dt$ for each value of u using the geometric formula for the area of a triangle and properties of integrals.

x	-2	-1	$-\frac{1}{3}$	0	$\frac{1}{3}$	1	2	3	
u	-6		-1	0		3			General Formula
$H(u)$									

(b) Graph the ordered pairs $(u, H(u))$ on the grid below.



(c) Write a quadratic function $H(u)$ that represents the graph from part (b). (It should match the formula from the chart above.)

$H(u) =$ ________________________

(d) What is $H'(u)$? __________________

(e) By substituting $u = 3x$, define $H(x) =$ __________________

(f) What is $H'(x)$? _____________________

Hence, if $H(x) = \int_a^{u=g(x)} f(t)\,dt$, where a is any real number,

then $H'(x) = f(u) \cdot \frac{du}{dx}$,

or $H'(x) = f(g(x)) \cdot g'(x)$.

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Worksheet 2. Applying the Second Fundamental Theorem of Calculus: Finding Derivatives of Functions Defined by an Integral

Teacher Notes for Worksheet 2

I use Worksheet 2 as an immediate follow-up to Worksheet 1. Students are to use the Second Fundamental Theorem of Calculus to complete the exercises.

Worksheet 2 also could be used to allow students who already know the First Fundamental Theorem of Calculus to discover the Second Fundamental Theorem of Calculus. For instance, in Exercise 6, these students would compute:

$$F(x) = \int_1^x (2 - 2t)dt = \left(2t - t^2\right)\Big|_1^x = 2x - x^2 - 1$$

using the First Fundamental Theorem of Calculus, then would compute the derivative as $F'(x) = 2 - 2x$.

Solutions to Worksheet 2

1. $F'(x) = 3$ (from Example 1 from Worksheet 1)
2. $F'(x) = 3$ (from Ex. 2)
3. $F'(x) = 2x$ (from Ex. 3)
4. $F'(x) = 2x$ (from Ex. 4)
5. $F'(x) = 2 - 2x$
6. $F'(x) = 2 - 2x$
7. $F'(x) = \sin x$
8. $F'(x) = \sin x$
9. $F'(x) = 2(3x) \cdot 3 = 18x$ (from Ex. 5)
10. $F'(x) = 2(\sin x) \cdot \cos x$
11. $F'(x) = f(x)$
12. $F'(x) = f(g(x))g'(x)$

The relationship between the integrand function and the integral function is that the integrand function (multiplied by the derivative of the upper limit of integration) is the derivative of the integral function. More specifically, the derivative of an integral function with upper limit of integration $g(x)$ is its integrand function, written in terms of *x,* and multiplied by the derivative of $g(x)$.

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Worksheet 2. Applying the Second Fundamental Theorem of Calculus: Finding Derivatives of Functions Defined by an Integral

Find $F'(x)$ when:

1. $F(x) = \int_0^x 3\,dt$
2. $F(x) = \int_1^x 3\,dt$
3. $F(x) = \int_0^x 2t\,dt$
4. $F(x) = \int_1^x 2t\,dt$
5. $F(x) = \int_0^x (2-2t)\,dt$
6. $F(x) = \int_1^x (2-2t)\,dt$
7. $F(x) = \int_0^x \sin t\,dt$
8. $F(x) = \int_1^x \sin t\,dt$
9. $F(x) = \int_0^{3x} 2t\,dt$
10. $F(x) = \int_0^{\sin x} 2t\,dt$
11. $F(x) = \int_a^x f(t)\,dt$
12. $F(x) = \int_a^{g(x)} f(t)\,dt$

What is the relationship between the integrand function and the integral function?

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Worksheet 3. Applying the First Fundamental Theorem of Calculus: Definite Integral as Total Change

Teacher Notes for Worksheet 3

After we derive the First Fundamental Theorem of Calculus, as shown at the top of Worksheet 3, my students and I thoroughly discuss its meaning in terms of physical applications, especially distance and velocity. Students can then practice their understanding of the definite integral of a rate of change as total change using the exercises from Worksheet 3. Regarding the derivation of the First Fundamental Theorem, students have learned that if $G(x)=\int_a^x f(t)\,dt$, then $G'(x)=f(x)$. Equivalently, $f(x)$ has as one of its antiderivatives $G(x)$. Letting $G(x)=F(x)+C$, where $F(x)$ is any antiderivative of $f(x)$ (assuming any two antiderivatives of $f(x)$ differ by a constant), we have the first line of Worksheet 3.

Solutions to Worksheet 3

1. The integral represents the change in the height of the cone, in feet, in the first five hours.
2. The integral represents the change in the position of the particle, in feet, from time t = 3 seconds to time t = 10 seconds.
3. The integral represents the change in the number of bacteria in the dish from time t = 2 hours to time t = 6 hours.
4. (a) $5+\int_2^{10} v(t)\,dt$ (b) $\int_2^{10} |v(t)|\,dt$

   Note to teachers: The integrand in part (b) is the speed of the particle at time *t*, $2\le t\le 10$. One way to see that $\int_2^{10}|v(t)|\,dt$ gives the total distance traveled by the particle is to note that the speed function $|v(t)|$, in transforming all negative velocities to positive velocities, also transforms all backward motion (motion to the left) to forward motion (motion to the right). The change in position from start to finish of the resulting forward-only journey is the total distance traveled during the original forward-and-backward journey.

5. $100+\int_0^2 p(t)\,dt$

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Worksheet 3. Applying the First Fundamental Theorem of Calculus: Definite Integral as Total Change

Derivation of the First Fundamental Theorem of Calculus

From Second FTC: $\int_a^x f(t)\,dt = F(x) + C$, where F is an antiderivative of f, and

$\int_a^a f(t)\,dt = F(a) + C = 0$, hence, $C = -F(a)$.

Let $x = b$: $\int_a^b f(t)\,dt = F(b) + C$, and, by substitution,

$\int_a^b f(t)\,dt = F(b) - F(a)$ or $F(b) = F(a) + \int_a^b f(t)\,dt$.

Accumulated (or total or net) change is given by the definite integral whose integrand is the rate of change. More specifically, if f is the rate of change of F, then:

$$\int_a^b f(t)dt = \text{Change in } F \text{ from } t = a \text{ to } t = b = F(b) - F(a).$$

Exercises

Write a sentence to answer each of the following questions.

1. If $h(t)$ is the rate of change of the height of a conical pile of sand measured in feet per hour, what does $\int_0^5 h(t)\,dt$ represent? Answer in correct units.

2. If $v(t)$ is the velocity of a particle moving along the x-axis, measured in feet per second, what does $\int_3^{10} v(t)\,dt$ represent? Answer in correct units.

3. If $b(t)$ is the rate of growth of the number of bacteria in a dish, measured in number of bacteria per hour, what does $\int_2^6 b(t)\,dt$ represent? Answer in correct units.

4. If $v(t)$ is the velocity of a particle moving along the *x*-axis at time *t*, and the position $x(t)$ is 5 at time $t = 2$, (a) write an integral expression that represents the position of the particle at time $t = 10$, and (b) write an integral expression that gives the total distance traveled by the particle from time $t = 2$ to time $t = 10$.

5. If $p(t)$ is the rate of growth of a rabbit population, measured in rabbits per year, and there were 100 rabbits in the year 2005 ($t = 0$), write an integral expression that represents the rabbit population in 2007.

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Worksheet 4. Example Multiple-Choice Questions

Teacher Notes for Worksheets 4 and 5

Worksheet 4 provides a brief review of the First and Second Fundamental Theorems of Calculus. Worksheet 5, written by my friend and colleague, Nancy Stephenson, of Clements High School in Sugar Land, Texas, provides a more extensive review of the First Fundamental Theorem of Calculus. These two worksheets can be used for review either at the end of the unit on this topic or just before students take the AP Exam.

Solutions to Worksheet 4

#23.(e)

#82.(a)

#91.(e)

#92.(d)

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Worksheet 4. Example Multiple-Choice Questions

#23. (2003 AB Exam, Section I, Part A, non-calculator section). $\frac{d}{dx}\left(\int_0^{x^2} \sin\left(t^3\right)dt\right) =$

(a) $-\cos(x^6)$ (b) $\sin(x^3)$ (c) $\sin(x^6)$ (d) $2x\sin(x^3)$ (e) $2x\sin(x^6)$

The following questions are from the 2003 AB Exam, Section I, Part B, calculator section.

#82. The rate of change of the altitude of a hot air balloon is given by:

$$r(t) = t^3 - 4t^2 + 6$$

for $0 \le t \le 8$. Which of the following expressions gives the change in altitude of the balloon during the time the altitude is decreasing?

(a) $\int_{1.572}^{3.514} r(t)dt$ (b) $\int_0^8 r(t)dt$ (c) $\int_0^{2.667} r(t)dt$ (d) $\int_{1.572}^{3.514} r'(t)dt$ (e) $\int_0^{2.667} r'(t)dt$

#91. A particle moves along the *x*-axis so that at any time $t > 0$, its acceleration is given by $a(t) = \ln(1+2^t)$. If the velocity of the particle is 2 at time $t = 1$, then the velocity of the particle at time $t = 2$ is

(a) 0.462 (b) 1.609 (c) 2.555 (d) 2.886 (e) 3.346

#92. Let *g* be the function given by:

$$g(x) = \int_0^x \sin\left(t^2\right)dt$$

for $-1 \le x \le 3$. On which of the following intervals is *g* decreasing?

(a) $-1 \le x \le 0$ (b) $0 \le x \le 1.772$ (c) $1.253 \le x \le 2.171$ (d) $1.772 \le x \le 2.507$ (e) $2.802 \le x \le 3$

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Worksheet 5. Applying the Fundamental Theorem of Calculus: Examples

Example 1

Given $\dfrac{dy}{dx} = 3x^2 + 4x - 5$ with the initial condition $y(2) = -1$, find $y(3)$.

Method 1

Find $y = \int (3x^2 + 4x - 5)dx$ by antidifferentiation, and use the initial condition to find C.

Then write the particular solution, and use the particular solution to find $y(3)$.

Solution

$$y = \int (3x^2 + 4x - 5)dx$$

$$y = x^3 + 2x^2 - 5x + C$$

$$-1 = 8 + 8 - 10 + C$$

$$-7 = C$$

$$y = x^3 + 2x^2 - 5x - 7$$

$$y(3) = 27 + 18 - 15 - 7 = 23$$

Method 2

Use the Fundamental Theorem of Calculus: $\int_a^b f'(x)dx = f(b) - f(a)$.

Solution

$$\int_2^3 y'dx = y(3) - y(2)$$

$$y(3) = y(2) + \int_2^3 y'dx$$

$$y(3) = -1 + \int_2^3 (3x^2 + 4x - 5)dx$$

$$y(3) = -1 + (x^3 + 2x^2 - 5x)\Big|_2^3$$

$$y(3) = -1 + (27 + 18 - 15) - (8 + 8 - 10)$$

$$y(3) = 23$$

Notice that $y(3) = 23$ by both methods. If a closed form of the antiderivative exists, either method may be used.

Sometimes there is no closed form of the antiderivative so we have to use Method 2 and a graphing calculator, as in Example 2.

Example 2

$f'(x) = \sin(x^2)$ and $f(2) = -5$. Find $f(1)$.

Solution

$$\int_1^2 f'(x)\,dx = f(2) - f(1)$$

$$f(1) = f(2) - \int_1^2 f'(x)\,dx$$

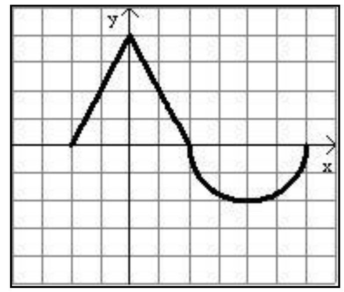
$$f(1) = -5 - \int_1^2 \sin(x^2)\,dx$$

$$f(1) = -5.495$$

Example 3

The graph of f' on $-2 \le x \le 6$ consists of two line segments and a semicircle as shown at right. Given that $f(-2) = 5$,

find $f(0)$, $f(2)$, and $f(6)$.



Graph of f'

Solution

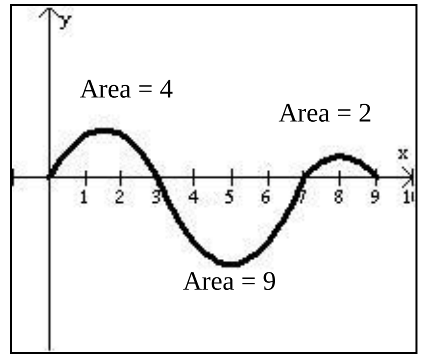
$$f(0) = f(-2) + \int_{-2}^{0} f'(x)\,dx = 5 + \frac{1}{2}(2)(4) = 9$$

$$f(2) = f(-2) + \int_{-2}^{2} f'(x)\,dx = 5 + \frac{1}{2}(4)(4) = 13$$

$$f(6) = f(-2) + \int_{-2}^{6} f'(x)\,dx = 5 + \frac{1}{2}(4)(4) - \frac{1}{2}\pi\left(2^2\right) = 13 - 2\pi$$

Example 4

The graph of f' is shown at right, with areas of regions enclosed by the graph and the *x*-axis as indicated.



Graph of f'

(a) Given that $f(3) = 5$, find $f(0)$, $f(7)$, and $f(9)$.

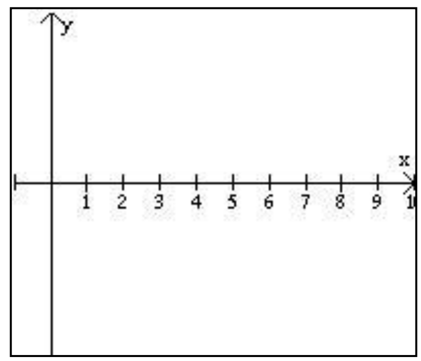
Solution

$$f(0) = f(3) - \int_0^3 f'(x)dx = 5 - 4 = 1$$

$$f(7) = f(3) + \int_3^7 f'(x)dx = 5 - 9 = -4$$

$$f(9) = f(3) + \int_3^9 f'(x)dx = 5 - 9 + 2 = -2$$

(b) Sketch the graph of *f*.



Graph of *f*

Solution

The graph of *f* should contain the points $(0,1)$, $(3, 5)$, $(7, -4)$, and $(9, -2)$ as local extrema, and should be concave up on $0 < x < 1.5$ (where f' is increasing), concave down on $1.5 < x < 5$ (where f' is decreasing), concave up on $5 < x < 8$ (where f' is increasing), and concave down on $8 < x < 9$ (where f' is decreasing).

Example 5

A pizza with a temperature of 95°C is put into a 25°C room when $t = 0$. The pizza's temperature is decreasing at a rate of $r(t) = 6e^{-0.1t}$ °C per minute. Estimate the pizza's temperature when $t = 5$ minutes.

Solution

Temperature = $95 - \int_0^5 6e^{-0.1t}\,dt = 71.392$°C

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Worksheet 5. Applying the Fundamental Theorem of Calculus: Exercises

Work Problems 1–3 by both methods.

1. $y' = 2 + \frac{1}{x^2}$ and $y(1) = 6$. Find $y(3)$.

2. $f'(x) = \cos(2x)$ and $f(0) = 3$. Find $f\left(\frac{\pi}{4}\right)$.

3. Water flows into a tank at a rate of $\frac{dW}{dt} = \frac{1}{75}\left(600 + 20t - t^2\right)$ where $\frac{dW}{dt}$ is measured in gallons per hour and t is measured in hours. If there are 150 gallons of water in the tank at time $t = 0$, how many gallons of water are in the tank when $t = 24$?

Work Problems 4–10 using the Fundamental Theorem of Calculus and your calculator.

4. $f'(x) = \cos\left(x^3\right)$ and $f(0) = 2$. Find $f(1)$.

5. $f'(x) = e^{-x^2}$ and $f(5) = 1$. Find $f(2)$.

6. A particle moving along the x-axis has position $x(t)$ at time t with the velocity of the particle given by $v(t) = 5\sin\left(t^2\right)$ At time $t = 6$, the particle's position is (4, 0). Find the position of the particle when $t = 7$.

7. Let $F(t)$ represent a bacteria population which is 4 million at time $t = 0$. After t hours, the population is growing at an instantaneous rate of 2^t million bacteria per hour. Find the total increase in the bacteria population during the first three hours, and find the population at $t = 3$ hours.

8. A particle moves along a line so that at any time $t \geq 0$ its velocity is given by $v(t) = \frac{t}{1+t^2}$. At time $t = 0$, the position of the particle is $s(0) = 5$. Determine the position of the particle at $t = 3$.

9. Let f be the function whose graph passes through the point (3, 6) and whose derivative is given by:

$$f'(x) = \frac{1+e^x}{x^2}. \text{ Find } f(3.1).$$

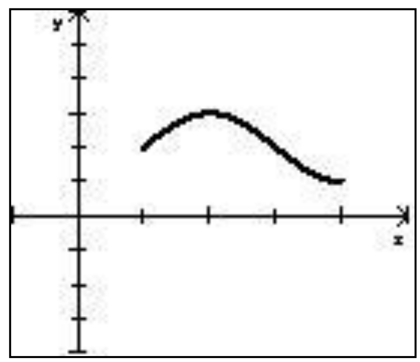
10. (Multiple Choice) If f is the antiderivative of $\frac{x^2}{1+x^5}$ such that $f(1)=5$, then $f(4)=$

(a) 4.988 (b) 5 (c) 5.016 (d) 5.376 (e) 5.629

In Problems 11–13, use the Fundamental Theorem of Calculus and the given graph. Each tick mark on the axes below represents one unit.

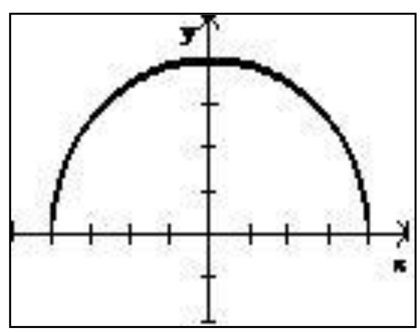
11. The graph of f' is shown at right.

$\int_1^4 f'(x)dx = 6.2$ and $f(1)=3$. Find $f(4)$.



12. The graph of f' is the semicircle shown at right.

Find $f(-4)$ given that $f(4)=7$.

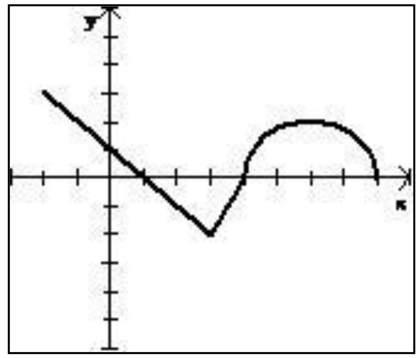


13. The graph of f', consisting of two line segments and a semicircle, is shown at right. Given that $f(-2)=5$, find:

(a) $f(1)$

(b) $f(4)$

(c) $f(8)$



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Worksheet 5. Answers

1. $\frac{32}{3}$
2. $\frac{7}{2}$
3. 357.36 gallons
4. 2.932
5. 0.996
6. 3.837
7. 10.099 million, 14.099 million
8. 6.151
9. 6.238
10. d
11. 9.2
12. $7-8\pi$
13. (a) 9.5 (b) 6.5 (c) $6.5 + 2\pi$

Nancy Stephenson teaches at Clements High School in Sugar Land, Texas. She was a member of the AP Calculus Examination Development Committee from 1999 to 2003 and is a College Board consultant.