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AP® Calculus:

Functions Defined by Integrals

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Curriculum Module

AP Calculus

Functions Defined by Integrals

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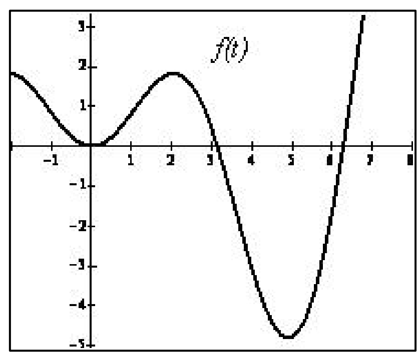
Reasoning from the graph of the derivative function *f* in order to obtain information about the behavior of the function *F* defined by $F(x)=\int_a^x f(t)\,dt$ has been a challenging task for students in many recent AP Examination questions. Since working with functions defined by integrals requires an understanding of the Second Fundamental Theorem of Calculus, this lesson begins with activities in which students discover and explore this theorem; it then moves on to its main theme of obtaining information about the function *F* from the graph of the derivative function *f*.

Introducing the Topic

Consider the following example. Let $F(x)=\int_2^x f(t)\,dt$, where the graph of $f(t)$ is shown at right. Notice that $F(x)$ is a function, but that it is not represented in a form familiar to students in their first calculus course. The example given is often overwhelming for a student for a variety of reasons: it is written in terms of an integral, it is difficult to keep straight the difference between *f* and *F*, and $f(t)$ is unknown. I know in my first attempts to solve problems with functions given in this representation I wished I had a formula for $f(t)$ in order to simplify the problem. But students should be able to do more with a function given in this form than reduce it to a simpler case. In the three worksheets that follow I hope to show ways to develop student understanding of functions defined by integrals by connecting students' prior calculus experience to this topic.



So what can calculus students do with functions by the time they reach their study of definite integrals? They should be able to find points on the graph of the function, certainly, and analyze the function using various derivatives, and in general think about the function using multiple representations. The challenge for AP Calculus teachers is to familiarize their students with the representation, $F(x)=\int_2^x f(t)\,dt$, and then build upon the connections between this representation and what the student already understands of calculus.

Prerequisites:

Students will need to:

- Understand the definite integral as signed area (before Worksheet 1).
- Understand the analysis of functions using first and second derivatives (before Worksheet 1).
- Understand the First Fundamental Theorem of Calculus (before Worksheet 2).
- Be open to this new representation of a function.

Timeline Suggestions

- I use Worksheet 1 after students first encounter the definite integral as signed area. It is intended to help students anticipate the formula for the derivative of a function defined as an integral; that is, the Second Fundamental Theorem of Calculus.
- I use Worksheet 2 after introducing the First Fundamental Theorem of Calculus in order to explore the Second Fundamental Theorem of Calculus.
- I use Worksheet 3 as a review of graphical analysis using the first and second derivatives of functions defined by integrals.

Worksheets and AP Examination Questions

Each of the worksheets includes additional notes for the instructor and complete solutions.

Problems featuring functions defined by integrals have occurred frequently on recent AP Calculus Examinations, including in the following free-response questions.

2007 AB 3, part (c)
2006 AB 3
2005 AB 4, parts (c) and (d)
2005 Form B, AB/BC 4
2004 AB 5
2003 Form B, AB/BC 5
2002 AB/BC 4
2002 Form B, AB/BC 4

Free-response questions from recent AP Calculus Examinations are available at AP Central (apcentral.collegeboard.com) at The AP Calculus AB Exam page or at The AP Calculus BC Exam page. From the AP Calculus AB Course Home Page, select Exam Information: The AP Calculus AB Exam; from the AP Calculus BC Course Home Page, select Exam Information: The AP Calculus BC Exam.

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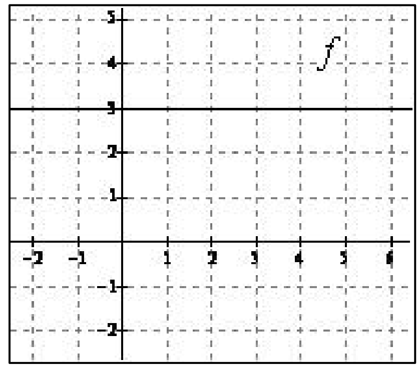
Worksheet 1. Exploring Functions Defined by Integrals

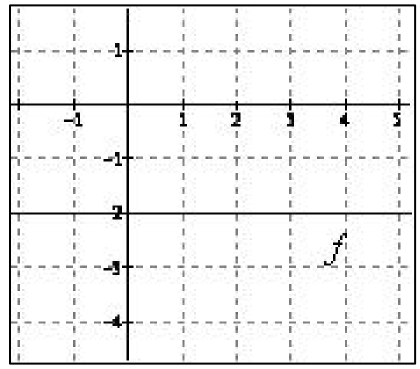
1. Let $F(x)=\int_0^x f(t)\,dt$, where the graph of f is shown at right. Complete the following chart.



x	-2	-1	0	1	2	3	4
$F(x)$							

On what interval is f positive?
On what interval, if any, is f negative?
On what interval is $F(x)$ increasing?
On what interval, if any, is $F(x)$ decreasing?
Determine an algebraic rule for f.
Determine an algebraic rule for F.

2. Let $F(x) = \int_2^x f(t)\,dt$, where the graph of *f* is shown at right. Complete the following chart.



x	-2	-1	0	1	2	3	4
$F(x)$							

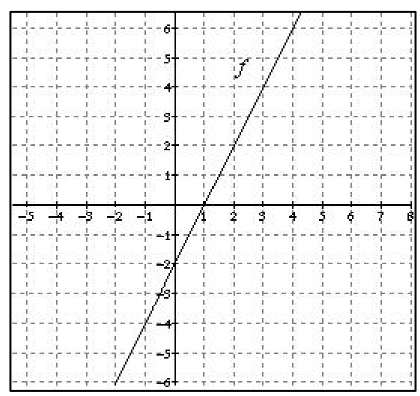
On what interval is *f* negative?

On what interval is $F(x)$ decreasing?

Determine an algebraic rule for *f*.

Determine an algebraic rule for *F*.

3. Let $F(x) = \int_0^x f(t)\,dt$, where the graph of f is shown at right. Complete the following chart.



x	-2	-1	0	1	2	3	4
$F(x)$							

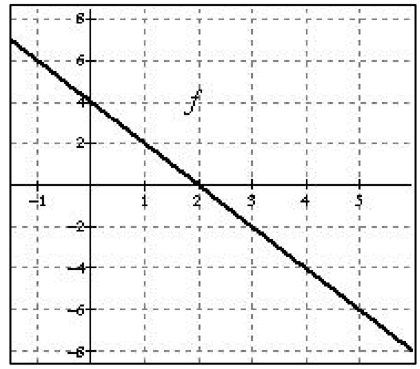
On what interval is *f* positive?

On what interval is *f* negative?

On what interval is *F* increasing?

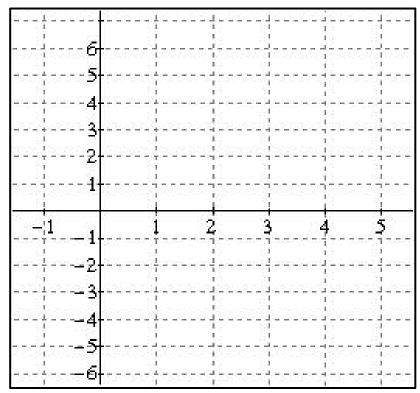
On what interval is *F* decreasing?

4. Let $F(x) = \int_0^x f(t)\,dt$, where the graph of f is shown at right. Complete the following chart.

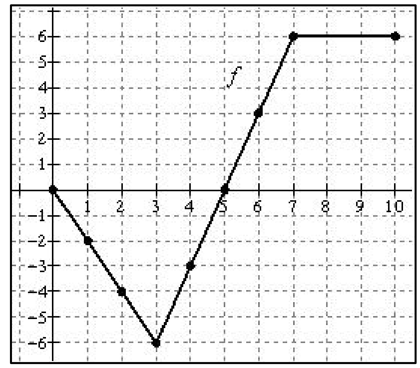


x	-1	0	1	2	3	4	5
$F(x)$							

On what interval is f positive?
On what interval is f negative?
On what interval is F increasing?
On what interval is F decreasing?
Plot the points $(x, F(x))$ on the grid provided.
What is the relationship between f and F?

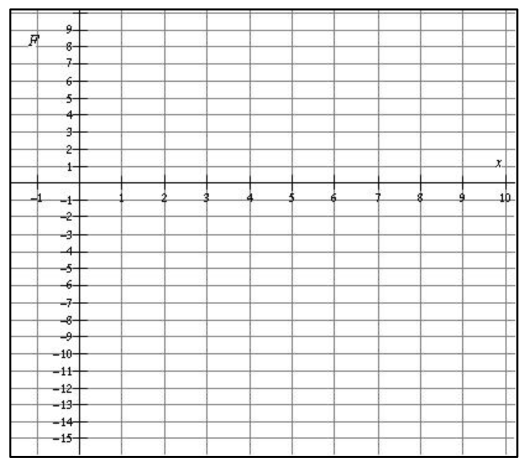


5. Let $F(x) = \int_0^x f(t)\,dt$, where the graph of f is shown at right. Complete the following chart.



x	0	1	2	3	4	5	6	7	8	9	10
$F(x)$											

On what interval is f positive?
On what interval is f negative?
On what interval is F increasing?
On what interval is F decreasing?



Plot the points $(x, F(x))$ on the grid provided.

On what interval is f decreasing?
On what interval is f increasing?
On what interval is the graph of F concave down?
On what interval is the graph of F concave up?
What is the relationship between f and F?

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Worksheet 1. Answers and Commentary

The intent of Worksheet 1 is for students to discover that when *f* is positive, $F(x)=\int_a^x f(t)\,dt$ is increasing, and when *f* is negative, $F(x)$ is decreasing, anticipating the relationship $F'(x)=f(x)$, or the Second Fundamental Theorem of Calculus for *f* continuous. In Exercise 5, they discover that when *f* is increasing, the graph of $F(x)$ is concave up, and when *f* is decreasing, the graph of $F(x)$ is concave down, providing further evidence for the relationship $F''=f'$ or $F'=f$.

Prerequisites include knowing how to analyze functions using their first and second derivatives and how to compute definite integrals as signed areas. More specifically, students should be able to calculate a definite integral as the signed area of the region(s) bounded by the graph of the integrand function, the *x*-axis, and the vertical lines given by the limits of integration — that is, as the sum of areas of regions above the *x*-axis, minus the sum of areas of regions below the *x*-axis. For example, the student should be able to calculate the definite integral:

$\int_{-2}^{5}(2-x)\,dx$ as $\int_{-2}^{5}(2-x)\,dx = 8-\frac{9}{2}=\frac{7}{2}$.

1.

x	-2	-1	0	1	2	3	4
$F(x)$	-6	-3	0	3	6	9	12

f is positive on the interval $(-\infty,\infty)$.

f is not negative.

F is increasing on the interval $(-\infty,\infty)$.

F does not decrease.

$f(t)=3$. $F(x)=3x$. Notice $F'=f$.

2.

x	-2	-1	0	1	2	3	4
$F(x)$	8	6	4	2	0	-2	-4

f is negative on the interval $(-\infty,\infty)$.

F is decreasing on the interval $(-\infty,\infty)$.

$f(t)=-2$. $F(x)=-2x+4$. Notice $F'=f$.

3.

x	-2	-1	0	1	2	3	4
$F(x)$	8	3	0	-1	0	3	8

f is positive on the interval $(1,\infty)$.

f is negative on the interval $(-\infty,1)$.

F is increasing on the interval $[1,\infty)$.

F is decreasing on the interval $(-\infty,1]$.

4.

x	-1	0	1	2	3	4	5
$F(x)$	-5	0	3	4	3	0	-5

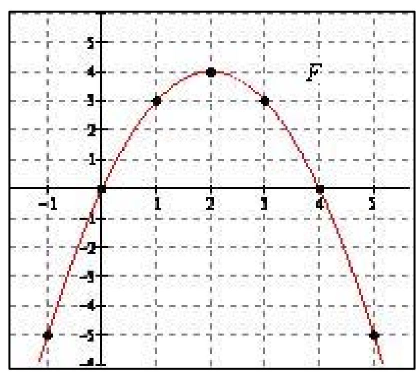
f is positive on the interval $(-\infty,2)$.

f is negative on the interval $(2,\infty)$.

F is increasing on the interval $(-\infty,2]$.

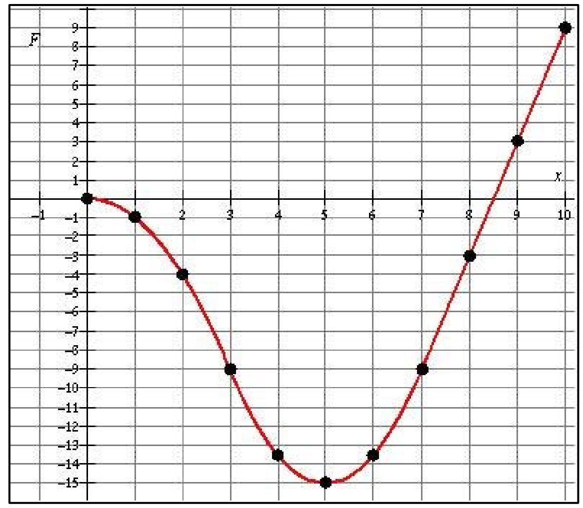
F is decreasing on the interval $[2,\infty)$.

The relationship between F and f is $F' = f$.



5.

x	0	1	2	3	4	5	6	7	8	9	10
$F(x)$	0	-1	-4	-9	-13.5	-15	-13.5	-9	-3	3	9



f is positive on the interval $(5,10]$.

f is negative on the interval $(0,5)$.

F is increasing on the interval $[5,10]$.

F is decreasing on the interval $[0,5]$.

f is (strictly) decreasing on the interval $[0,3]$.

f is (strictly) increasing on the interval $[3,7]$.

The graph of F is concave down on the interval $(0,3)$.

The graph of F is concave up on the interval $(3,7)$.

$F' = f$ and/or $F'' = f'$.

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Worksheet 2. Exploring Derivatives of Functions Defined by Integrals

Second Fundamental Theorem of Calculus:

If $F(x)=\int_a^x f(t)\,dt$, where *a* is constant and *f* is a continuous function, then:

$$F'(x)=f(x).$$

If $F(x)=\int_a^{g(x)} f(t)\,dt$, where *a* is constant, *f* is a continuous function, and *g* is a differentiable function, then:

$$F'(x)=f\big(g(x)\big)\cdot g'(x).$$

Check this result by performing the following two steps for each of the functions in Exercises 1–3.

Step One:

Given $f(t)$, evaluate $F(x)=\int_1^x f(t)\,dt$ to find $F(x)$ in terms of *x*.

Step Two:

Take the derivative of the result to determine $F'(x)$.

1. $f(t)=t^3$

2. $f(t)=4t-t^2$

3. $f(t)=\cos(t)$

Let us consider the even more difficult case of $F(x) = \int_1^{x^2} f(t)\,dt$. Check the result for this case by performing the following two steps for each of the functions in Exercises 4–6.

Step One:

Given $f(t)$, evaluate $F(x) = \int_1^{x^2} f(t)\,dt$ to find $F(x)$ in terms of x.

Step Two:

Take the derivative of the result to determine $F'(x)$.

4. $f(t) = t^3$

5. $f(t) = \cos(t)$

6. $f(t) = 6\sqrt{t}$

7. If $F(x) = \int_3^{x^2} \tan(t)\,dt$, then $F'(x) =$ ____________________

8. If $F(x) = \int_3^{g(x)} \tan(t)\,dt$, then $F'(x) =$ ____________________

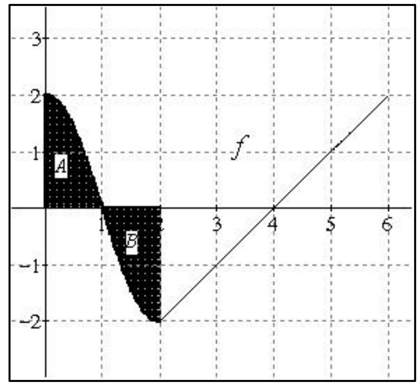
9. If $F(x) = \int_1^{2x} f(t)\,dt$, then $F'(x) =$ ____________________

10. Let $H(x) = \int_{\pi/2}^{x} t\cos(t)\,dt$ for $0 < x < 2\pi$.
    (a) Determine the critical numbers of $H(x)$.

    (b) Determine which critical number corresponds to a relative maximum value of $H(x)$. Justify your answer.

    (c) Determine which critical number corresponds to a relative minimum value of $H(x)$. Justify your answer.

11. Let $F(x)=\int_1^{2x} f(t)\,dt$, where the graph of f on the interval $0 \le t \le 6$ is shown at right, and the regions A and B each have an area of 1.3.



(a) Compute $F(0)$ and $F(1)$.

(b) Determine $F'(x)$.

(c) Determine the critical numbers of $F(x)$ on the interval $0 \le x \le 3$.

(d) Determine which critical number of $F(x)$ corresponds to a relative maximum value of $F(x)$ on the interval $0 \le x \le 3$. Justify your answer.

(e) Determine which critical number of $F(x)$ corresponds to a relative minimum value of $F(x)$ on the interval $0 \le x \le 3$. Justify your answer.

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Worksheet 2. Answers and Commentary

This worksheet is intended to help convince students of the validity of the Second Fundamental Theorem of Calculus and to give them practice in using it. The main prerequisite is to know how to use the First Fundamental Theorem of Calculus. For Exercises 10 and 11, students also will need to know how to analyze a function using its first derivative.

1. Step one: $F(x) = \int_1^x t^3\,dt = \frac{1}{4}t^4\Big|_1^x = \frac{1}{4}x^4 - \frac{1}{4}$. Step two: $F'(x) = x^3$.

2. Step one: $F(x) = \int_1^x \left(4t - t^2\right)dt = \left(2t^2 - \frac{1}{3}t^3\right)\Big|_1^x = 2x^2 - \frac{1}{3}x^3 - \left(2 - \frac{1}{3}\right)$

   Step two: $F'(x) = 4x - x^2$

3. Step one: $F(x) = \int_1^x \cos t\,dt = \sin t\,\Big|_1^x = \sin(x) - \sin(1)$

   Step two: $F'(x) = \cos x$

4. Step one: $F(x) = \int_1^{x^2} t^3\,dt = \frac{1}{4}t^4\Big|_1^{x^2} = \frac{1}{4}\left(x^2\right)^4 - \frac{1}{4}(1)^4$

   Step two: $F'(x) = \left(x^2\right)^3 \cdot 2x = 2x^7$

5. Step one: $F(x) = \int_1^{x^2} \cos t\,dt = \sin t\,\Big|_1^{x^2} = \sin\left(x^2\right) - \sin(1)$

   Step two: $F'(x) = \cos\left(x^2\right) \cdot 2x$

6. Step one: $F(x) = \int_1^{x^2} 6\sqrt{t}\,dt = 4t^{3/2}\Big|_1^{x^2} = 4\left(x^2\right)^{3/2} - 4$

   Step two: $F'(x) = 6\sqrt{x^2} \cdot 2x = 12x^2$

7. $F'(x) = \tan\left(x^2\right) \cdot 2x$

8. $F'(x) = \tan\left(g(x)\right) \cdot g'(x)$

9. $F'(x) = f(2x) \cdot 2$

10. (a) $H'(x) = x\cos x = 0$ when $x = \dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$.

(b) At $x = \dfrac{\pi}{2}$, $H(x)$ has a relative maximum because $H'(x)$ changes from positive to negative.

(c) At $x = \dfrac{3\pi}{2}$, $H(x)$ has a relative minimum because $H'(x)$ changes from negative to positive.

11. (a) $F(0) = \int_1^0 f(t)\,dt = -\int_0^1 f(t)\,dt = -1.3$, while $F(1) = \int_1^2 f(t)\,dt = -1.3$

(b) $F'(x) = f(2x) \cdot 2$

(c) $F'(x) = f(2x) \cdot 2 = 0$ when $x = \dfrac{1}{2}$ and $x = 2$.

(d) At $x = \dfrac{1}{2}$, $F(x)$ has a relative maximum because $F'(x) = 2f(2x)$ changes from positive to negative.

(e) At $x = 2$, $F(x)$ has a relative minimum because $F'(x) = 2f(2x)$ changes from negative to positive.

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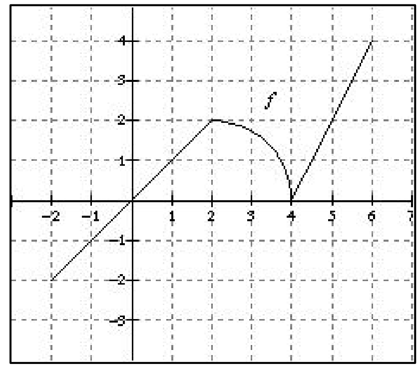
Worksheet 3. Graphical Analysis of $F(x)$ Using $F'(x)$

1. Let $F(x) = \int_2^x f(t)\,dt$. The graph of *f* on the interval $[-2, 6]$ consists of two line segments and a quarter of a circle, as shown at right.



   (a) Find $F(0)$ and $F(4)$.

   (b) Determine the interval where $F(x)$ is increasing. Justify your answer.

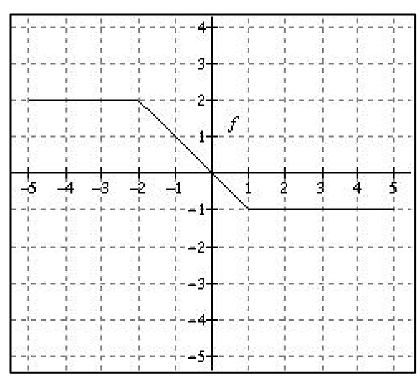
   (c) Find the critical numbers of $F(x)$ and determine if each corresponds to a relative minimum value, a relative maximum value, or neither. Justify your answers.

   (d) Find the absolute extreme values of $F(x)$ and the *x*-values at which they occur. Justify your answers.

   (e) Find the *x*-coordinates of the inflection points of $F(x)$. Justify your answer.

   (f) Determine the intervals where the graph of $F(x)$ is concave down. Justify your answer.

2. Let $H(x) = \int_0^{x+2} f(t)\,dt$, where *f* is defined on the interval $[-5, 5]$ and the graph of *f* consists of three line segments, as shown at right.

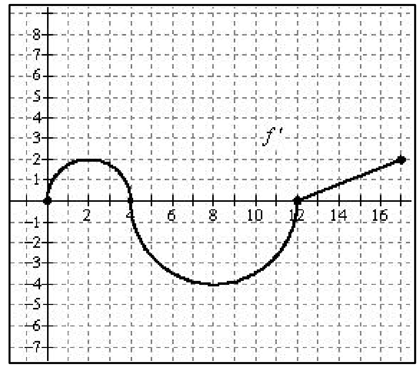


   (a) Determine the domain of $H(x)$.

   (b) Determine the range of $H(x)$.

   (c) Determine the *x*-coordinates of the relative extrema of $H(x)$. Justify your answer.

3. The graph below is of the function $f'(x)$, the derivative of the function $f(x)$, on the interval $0 \le x \le 17$. The graph consists of two semicircles and one line segment. Horizontal tangents are located at $x = 2$ and $x = 8$, and a vertical tangent is located at $x = 4$.



(a) On what intervals is $f(x)$ increasing? Justify your answer.

(b) For what values of *x* does $f(x)$ have a relative minimum value? Justify.

(c) On what intervals, for $0 < x < 17$, is the graph of $f(x)$ concave up? Justify.

(d) For what values of *x,* for $0 < x < 17$, is $f''(x)$ undefined?

(e) Identify the *x*-coordinates of all points of inflection of $f(x)$. Justify.

(f) For what value of *x* does $f(x)$ reach its absolute maximum value? Justify.

(g) If $f(4) = 3$, find $f(12)$.

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Worksheet 3. Answers and Commentary

This worksheet is intended to give students practice in graphical analysis using the first and second derivatives of functions defined by integrals. Note that in Exercises 1 and 2, the functions *F* and *H* are defined by integrals, whereas in Exercise 3, the function *f* is not explicitly defined as an integral. Exercise 3 is intended to remind students that the process of obtaining information about a function from the graph of its derivative is the same, whether or not the function is defined as an integral. You may wish to use Exercise 3 as a warm-up for Exercises 1 and 2.

For further practice, be sure to have students work the free-response questions from recent AP Calculus Examinations listed in the introduction to this lesson. These are available at AP Central (apcentral.collegeboard.com).

Prerequisites for this worksheet include understanding of and experience with analysis of functions using first and second derivatives, the definite integral as signed area, and both parts of the Fundamental Theorem of Calculus.

1. (a) $F(0)=\int_2^0 f(t)\,dt=-\int_0^2 f(t)\,dt=-2$

   $F(4)=\int_2^4 f(t)\,dt=\pi$

   (b) $F(x)$ is increasing when $F'(x)$ is positive.
   Since $F'(x)=f(x)$ and $f(x)$ is positive on the intervals (0,4) and (4,6), then $F(x)$ is increasing on these intervals and in fact on the interval [0,6].

   (c) At $x = 0$, $F(x)$ has a relative minimum because $F'=f$ changes from negative to positive there.
   At $x = 4$, $F(x)$ has neither a relative minimum nor a relative maximum because $F'=f$ does not change sign there.

   (d) Using the Candidates Test, we compare $F(-2)=0$, $F(0)=-2$, $F(4)=\pi$, and $F(6)=\pi+4$, and find that $F(0)=-2$ is the absolute minimum value of *F* on the interval $[-2,6]$ and $F(6)=\pi+4$ is the absolute maximum value of *F* on the interval $[-2,6]$.

   (e) $F(x)$ has inflection points at $x = 2$ and $x = 4$ because $F''=f'$ changes signs at these points.

(f) The graph of $F(x)$ is concave down on the interval (2, 4) because $F'(x) = f(x)$ is strictly decreasing on this interval and $F''(x) = f'(x)$ is defined there, or, equivalently, because $F''(x)$ is negative on the interval.

2. (a.) Domain: $[-7, 3]$

   Since f is defined on the interval $[-5, 5]$, then $-5 \le x + 2 \le 5$, or $-7 \le x \le 3$. Another suitable method for determining the domain is to guess and check x-values for which the integral can be found.

   (b) Range: $[-8, 0]$. In order to determine the range, find the absolute extreme values of $H(x)$. Note that $x = -2$ is a critical number because $H'(x) = f(x+2) = 0$ when $x + 2 = 0$ or $x = -2$. The candidates for the absolute extrema are $H(-7)$, $H(-2)$, and $H(3)$.

   $H(-7) = \int_0^{-5} f(t)\,dt = -8$ is the absolute minimum value of H.

   $H(-2) = \int_0^{0} f(t)\,dt = 0$ is the absolute maximum value of H.

   $H(3) = \int_0^{5} f(t)\,dt = -4.5$

   (c) There is a relative (and absolute) maximum value of 0 at $x = -2$ because $H'(x) = f(x+2)$ changes from positive to negative at $x = -2$. There are relative minimum values of -8 at $x = -7$ and -4.5 at $x = 3$ because $H'(x) = f(x+2)$ is positive for $-7 < x < -2$ and negative for $-2 < x < 3$.

3. (a) Since f' is positive on the intervals (0,4) and (12,17], f is increasing on these intervals and in fact on the intervals [0, 4] and [12, 17].

   (b) At $x = 12$, f has a relative minimum because f' changes from negative to positive there. Note that f also has a relative minimum at $x = 0$ since f' is positive on the interval (0,4).

   (c) The graph of f is concave up on the intervals (0, 2), (8, 12) and (12, 17) because f' is strictly increasing there and f'' is defined there, or, equivalently, because f'' is positive there.

   (d) The slope of the graph of f' is not defined at $x = 4$ and $x = 12$.

   (e) There are inflection points at $x = 2$ and $x = 8$ because the slope of the graph of f' changes signs there. That is, f'' changes signs there.

   (f) $f(4)$ is the maximum value of f. The candidates are $x = 4$ and $x = 17$. By the Candidates Test, if $f(0) = C$, then $f(4) = C + 2\pi$ and

$f(17) = C + 2\pi - 8\pi + 5 = C - 6\pi + 5$. Since $f(17) - f(4) = -8\pi + 5 < 0$, then $f(4) > f(17)$.

(g) By the Fundamental Theorem of Calculus, $\int_4^{12} f'(x)\,dx = f(12) - f(4)$.
By substitution, $-8\pi = f(12) - 3$; therefore, $f(12) = 3 - 8\pi$.

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